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О СПЕКТРАЛЬНОЙ НОРМЕ ПРОРЕЖЕННЫХ ПРЯМОУГОЛЬНЫХ СЛУЧАЙНЫХ МАТРИЦ

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Аннотация

Мы доказываем верхнюю оценку для спектральной нормы $m \times n$ ($n \leqslant m$) прямоугольной случайной матрицы в предположении, что распределение ее элементов имеет конечный момент порядка $4+\delta$ и элементы матрицы усечены на уровне $(np)^{1/2-\varkappa}$, где $\varkappa > 0$ и зависит от δ. Символ p означает вероятность прореживания.

Ключевые слова:
выборочная ковариационная матрица, прореженная матрица, наибольшее сингулярное число, случайная матрица

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ON THE SPECTRAL NORM OF SPARSE RECTANGULAR RANDOM MATRICES

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Abstract

We prove an upper bound for the spectral norm of $n \times m$ ($n \leqslant m$) of a rectangular random matrix under the assumption that the distribution of matrix elements has a finite moment of order $4+\delta$ ($\delta > 0$) and the elements are truncated at the level $(np)^{1/2-\varkappa}$, where $\varkappa > 0$ and depends on δ. Here p denotes the probability of sparsity.

Keywords:
sample covariance matrix, sparse matrix, largest singular value, random matrix

Introduction and the main result

Let $m = m(n)$, $m \geqslant n$. Consider independent zero mean random variables X_{jk}, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant m$ with $\mathbb{E}X_{jk}^2 = 1$ and independent of that Bernoulli random variables ξ_{jk}, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant m$ with $\mathbb{E}\xi_{jk} = p_n$. In addition suppose that $np_n \to \infty$ as $n \to \infty$. Then we put $p = p_n$. Consider the sequence of random matrices

$$\mathbf{X} = (\xi_{jk}X_{jk})_{1\leqslant j\leqslant n, 1\leqslant k\leqslant m}. \tag{1}$$

We denote the non-zero singular values of the matrix $\mathbf{X}$ by $s_1 \geqslant \cdots \geqslant s_n$ and define the sample covariance matrix $\mathbf{W} = \mathbf{X}\mathbf{X}^*$.

Let $y = y(n) = \frac{n}{m}$. We are interesed in estimating the maximal singular value s_1 (or spectral norm) of matrix $\mathbf{X}$. The problem of estimating the norm of a random matrix arises in many applications. This topic has been studied by many authors. The case of Wigner matrices or sample covariance matrices has been sufficiently studied. See, for example, the work of Rudelson and Vershynin [1] and the literature to it. Sparse matrices with $p_n \to 0$ when $n \to \infty$ take a special place. Our motivation for studing the largest singular value of a sparse random matrix is related to the proof of local laws for sparse covariance matrices with "heavy tailed" entries.

The main result of our paper is the following

Theorem 1. *Let $\mathbb{E}X_{jk} = 0$ and $\mathbb{E}|X_{jk}|^2 = 1$. Let's assume that*

$$\mathbb{E}|X_{jk}|^{4+\delta} \leqslant C_0 < \infty,$$

for any $j, k \geqslant 1$ and for some $\delta > 0$. Suppose that there exists a positive constant B, such that

$$np_n \geqslant B \log^{\frac{2}{\kappa}} n,$$

where $\varkappa = \frac{\delta}{2(4+\delta)}$. Additionally assume that

$$|X_{jk}| \leqslant C_1(np_n)^{\frac{1}{2}-\kappa}. \tag{2}$$

Then for every $Q \geqslant 1$ there exists a constant $C = C(Q, \delta, C_0, C_1)$, such that

$$\Pr\{ s_1 \geqslant C\sqrt{np}\sqrt{\log n}\} \leqslant Cn^{-Q}, \quad (3)$$

The estimations of the largest singular value for sparse covariance matrices like (3) but without factor $\sqrt{\log n}$, with stricter restrictions of moments, is possible to find in [2].

Some applications and proof of the main result

To estimate the spectral norm of the matrix $\mathbf{X}$, we introduce the matrix $\mathbf{V}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{O} & \mathbf{X} \\ \mathbf{X}^* & \mathbf{O} \end{bmatrix}, \quad (4)$$

where $\mathbf{O}$ denotes a matrix of the corresponding dimension with zero entries. Note that

$$\mathbf{V}^2 = \begin{bmatrix} \mathbf{X}\mathbf{X}^* & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^*\mathbf{X} \end{bmatrix}$$

and

$$Tr\mathbf{W}^s = 2Tr\mathbf{V}^{2s}.$$

Let's consider the estimates of the high order moments of the matrix $\mathbf{V}$. First we note that for $q = 2r + 1$

$$\mathbb{E}Tr\mathbf{V}^q = 0.$$

We investigate $q = 2r$. Let $\alpha_{2k} = \frac{1}{n}Tr\mathbf{V}^{2k}$.

Theorem 2. *The following inequality holds for $r \leqslant C(np)^{\varkappa}$,*

$$\mathbb{E}\alpha_{2r} \leqslant np\sum_{s=0}^{[\frac{r-1}{2}]}\mathbb{E}\alpha_{2s}\alpha_{2r-2s-2}+$$

$$+Cnp\frac{r}{(np)^{\varkappa}}\mathbb{E}^{\frac{r-1}{r}}\alpha_{2r} + C\mathbb{E}^{\frac{2r-1}{2r}}\alpha_{2r}.$$

Corollary 1. *Under conditions of Theorem 2, for $r \leqslant C_0(np)^{\varkappa}$ there exist the constant $C_1 > 0$ depending on C_0 such that the following inequality holds*

$$\mathbb{E}\alpha_{2r} \leqslant C_1^r r^r (np)^r$$

Proof of Corollary 1 . Let

$$y_r = (np)^{-\frac{1}{2}}\mathbb{E}^{\frac{1}{2r}}\alpha_{2r}.$$

Using Young inequality, we can rewrite the result of Theorem 2 as follows

$$y_r^{2r} \leqslant \frac{1}{r}\left[4r(1+\frac{1}{(np)^{\varkappa}})\right]^r +$$

$$+\frac{1}{4}y_r^{2r} + \frac{1}{r}(4C)^{2r} + \frac{1}{4}y_r^{2r}.$$

Here from we get the required.

□

Corollary 2. *There exists an absolute constant C, s.t. for every $t > C$,*

$$\Pr\{s_1 \geqslant t\sqrt{np}\sqrt{\log n}\} \leqslant \exp\{-c\log t\log n\}.$$

Proof of Corollary 2. Note that $(np)^{\varkappa} > C\log n$. We put $r = c\log n$. It is easy to see that

$$s_1 \leqslant n^{\frac{1}{2r}}\alpha_{2r}^{\frac{1}{2r}}.$$

Applying Chebyshev's inequality, we get

$$\Pr\{s_1 \geqslant t\sqrt{np}\sqrt{\log n}\} \leqslant \frac{\mathbb{E}s_1^{2r}}{t^{2r}(np)^r\log^r n} \leqslant$$

$$\leqslant \frac{n\mathbb{E}\alpha_{2r}}{t^{2r}(np)^r\log n^r}.$$

Using Theorem 2, we get

$$\Pr\{s_1 \geqslant t\sqrt{np}\sqrt{\log n}\} \leqslant \frac{nC^r}{t^{2r}} \leqslant \left(\frac{C}{t}\right)^{2r}.$$

□

Lemma 1.

$$\mathbb{E}Tr\mathbf{V}^{2r} = \sum_{s=0}^{[\frac{r-1}{2}]}(A_s + B_s) + Z_r$$

where

$$A_s = \sum_{j=1}^{n}\mathbb{E}\Big[\sum_{k=1}^{m}\sum_{l=1}^{m}X_{jk}\xi_{jk}X_{jl}\xi_{jl}\times$$

$$\times\left[(\mathbf{V}^{(j)})^{2s}\right]_{k+n,l+n}\Big]\left[\mathbf{V}^{2r-2s-2}\right]_{jj},$$

$$B_s = \sum_{j=1}^{n}\mathbb{E}\Big[\sum_{k=1}^{m}X_{jk}\xi_{jk}\left[(\mathbf{V}^{(j)})^{2s+1}\right]_{k+n,j}\Big]\times$$

$$\times\left[\mathbf{V}^{2r-2s-2}\right]_{j,j},$$

$$Z_r = \sum_{j=1}^{n}\sum_{k=1}^{m}\mathbb{E}X_{jk}\xi_{jk}\left[(\mathbf{V}^{(j)})^{r-1}\mathbf{V}^r\right]_{k+n,j}.$$

Proof of Lemma 1. We have the following equality

$$\mathbb{E}Tr\mathbf{V}^{2r} = 2\sum_{j=1}^{n}\sum_{k=1}^{m}\mathbb{E}X_{jk}\xi_{jk}\left[\mathbf{V}^{2r-1}\right]_{k+n,j}$$

Let's denote the jth column of the matrix $\mathbf{V}$ for $j = 1, \ldots, n$ by $\mathbf{V}_j$. Let

$$\mathbf{\Delta}_j = \mathbf{V}_j\mathbf{e}_j^T + \mathbf{e}_j\mathbf{V}_j^T - \frac{1}{\sqrt{mp}}X_{jj}\mathbf{e}_j\mathbf{e}_j^T$$

and

$$\mathbf{V}^{(j)} = \mathbf{V} - \mathbf{\Delta}_j. \quad (5)$$

Using these notations, we obtain

$$\mathbb{E}Tr\mathbf{V}^{2r} = C_1 + C_2,$$

where

$$C_1 = \sum_{j=1}^{n}\sum_{k=1}^{m}\mathbb{E}X_{jk}\xi_{jk}\left[\mathbf{V}^{(j)}\mathbf{V}^{2r-2}\right]_{k+n,j},$$

$$C_2 = \sum_{j=1}^{n}\sum_{k=1}^{m}\mathbb{E}X_{jk}\xi_{jk}\left[\mathbf{\Delta}_j\mathbf{V}^{2r-2}\right]_{k+n,j}.$$

It is straightforward to see that

$$C_2 = \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}X_{jk}^2\xi_{jk}\left[\mathbf{V}^{2r-2}\right]_{j,j} = A_0.$$

We continue with C_1. Using representation (5), we get

$$C_1 = C_{11} + C_{12},$$

where

$$C_{11} = \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}X_{jk}\xi_{jk}\left[(\mathbf{V}^{(j)})^2\mathbf{V}^{2r-3}\right]_{k+n,j},$$

$$C_{12} = \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}X_{jk}\xi_{jk}\left[\mathbf{V}^{(j)}\boldsymbol{\Delta}_j\mathbf{V}^{2r-3}\right]_{k+n,j}$$

Next, it is easy to check that

$$C_{12} = \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}X_{jk}\xi_{jk}\left[\mathbf{V}^{(j)}\right]_{k+n,j}\left[\mathbf{V}^{2r-2}\right]_{j,j} = B_0.$$

Note that

$$C_{12} = 0.$$

We continue with C_{11}. Applying again representation (5), we get

$$C_{11} = C_{111} + C_{112},$$

where

$$C_{111} = \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}X_{jk}\xi_{jk}\left[(\mathbf{V}^{(j)})^3\mathbf{V}^{2r-4}\right]_{k+n,j},$$

$$C_{112} = \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}X_{jk}\xi_{jk}\left[(\mathbf{V}^{(j)})^2\boldsymbol{\Delta}_j\mathbf{V}^{2r-4}\right]_{k+n,j}.$$

Using the definition of $\boldsymbol{\Delta}_j$, we get

$$C_{112} = \sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} \mathbb{E}X_{jk}X_{jl}\xi_{jk}\xi_{jl}\left[(\mathbf{V}^{(j)})^2\right]_{k+n,l+n}\times$$

$$\times\left[\mathbf{V}^{2r-4}\right]_{j,j} = A_2.$$

Repeating with C_{111}, we get

$$C_{111} = B_1 + \sum_{j=1}^{n}\sum_{k=1}^{m} \mathbb{E}X_{jk}\xi_{jk}\left[(\mathbf{V}^{(j)})^4\mathbf{V}^{2r-5}\right]_{k+n,j}.$$

Continuing this procedure, we get the required. □

Proof of Theorem 2. We start with the estimation of A_s, for $s = 1, \ldots, \left[\frac{r-1}{2}\right]$. We represent

$$A_s = \widetilde{A}_s + \widehat{A}_s,$$

where

$$\widetilde{A}_s = \sum_{j=1}^{n}\mathbb{E}\Big[\sum_{k=1}^{m} X_{jk}^2\xi_{jk}[(\mathbf{V}^{(j)})^{2s}]_{k+n,k+n}\Big]\times$$

$$\times\left[\mathbf{V}^{2r-2s-2}\right]_{jj},$$

$$\widehat{A}_s = \sum_{j=1}^{n}\mathbb{E}\Big[\sum_{k=1}^{m}\sum_{l=1,l\neq k}^{m} X_{jk}\xi_{jk}X_{jl}\xi_{jl}\times$$

$$\times\left[(\mathbf{V}^{(j)})^{2s}\right]_{k+n,l+n}\Big]\left[\mathbf{V}^{2r-2s-2}\right]_{jj}.$$

Further, we continue with $\widetilde{A}_s$ as follows

$$\widetilde{A}_s = \widetilde{A}_s^{(1)} + \widetilde{A}_s^{(2)},$$

where

$$\widetilde{A}_s^{(1)} = p\sum_{j=1}^{n}\mathbb{E}\Big[\sum_{k=1}^{m}\left[(\mathbf{V}^{(j)})^{2s}\right]_{k+n,k+n}\Big]\times$$

$$\times\left[\mathbf{V}^{2r-2s-2}\right]_{jj},$$

$$\widetilde{A}_s^{(2)} = \sum_{j=1}^{n}\mathbb{E}\Big[\sum_{k=1}^{m}(X_{jk}^2\xi_{jk} - p)\times$$

$$\times\left[(\mathbf{V}^{(j)})^{2s}\right]_{k+n,k+n}\Big]\left[\mathbf{V}^{2r-2s-2}\right]_{jj}.$$

Using $\mathbf{V}_{kk} \geqslant 0$ and $\mathbf{V}_{kk}^{(j)} \geqslant 0$ for $k = 1, \ldots, n+m$ and interplacing theorem, we get

$$\widetilde{A}_s^{(1)} \leqslant p\mathbb{E}Tr\mathbf{V}^{2s}Tr\mathbf{V}^{2r-2s-2}.$$

We can rewrite it as

$$\frac{1}{n}\widetilde{A}_s^{(1)} \leqslant np\mathbb{E}\alpha_{2s}\alpha_{2r-2s-2}.$$

Applying Hölder inequality, we get

$$|\widetilde{A}_s^{(2)}| \leqslant \sum_{j=1}^{n}\mathbb{E}^{\frac{s-1}{r-1}}\Big|\sum_{k=1}^{m}(X_{jk}^2\xi_{jk} - \mathbb{E}X_{jk}^2\xi_{jk})\times$$

$$\times\left[(\mathbf{V}^{(j)})^{2s}\right]_{k+n,k+n}\Big|^{\frac{r-1}{s-1}}\times$$

$$\mathbb{E}^{\frac{r-s-1}{r-1}}\Big|\left[\mathbf{V}^{2r-2s-2}\right]_{jj}\Big|^{\frac{r-s-1}{r-1}}.$$

Note that for $s = 0, \ldots, [\frac{r-1}{2}]$ we have $q = \frac{r-1}{s} \geqslant 2$. Taking the conditional expectation with respect to $\mathbf{V}^{(j)}$ and applying Rosenthal's inequality, we get the inequality

$$|\widetilde{A}_s^{(2)}| \leqslant \sum_{j=1}^{n}\mathbb{E}^{\frac{1}{q}}\Big(\Big(qp\sum_{k=1}^{m}(\left[(\mathbf{V}^{(j)})^{\frac{2(r-1)}{q}}\right]_{k+n,k+n}\Big)^2\Big)^{\frac{q}{2}} +$$

$$+q^q p(np)^{q-2q\varkappa-2}\sum_{k=1}^{m}\Big(\left[(\mathbf{V}^{(j)})^{\frac{2(r-1)}{q}}\right]_{k+n,k+n}\Big)^q\Big)\times$$

$$\times\mathbb{E}^{\frac{q-1}{q}}\Big|\left[\mathbf{V}^{2(r-1)\frac{q}{q-1}}\right]_{jj}\Big|^{\frac{q}{q-1}}.$$

This implies that

$$|\widetilde{A}_s^{(2)}| \leqslant \Gamma_s^{(1)} + \Gamma_s^{(2)},$$

where

$$\Gamma_s^{(1)} = C\sqrt{2npq}\sum_{j=1}^n \mathbb{E}^{\frac{1}{q}}\Big(\frac{1}{2n}Tr\left(\mathbf{V}^{(j)}\right)^{2(r-1)}\Big)\times$$

$$\times\mathbb{E}^{\frac{q-1}{q}}\Big|\Big[\mathbf{V}^{2(r-1)\frac{q}{q-1}}\Big]_{jj}\Big|^{\frac{q}{q-1}},$$

$$\Gamma_s^{(2)} = q(np)^{1-2\varkappa-1/q}n^{-\frac{1}{q}}\times$$

$$\times\sum_{j=1}^n \mathbb{E}^{\frac{1}{q}}\Big(\sum_{k=1}^m\Big(\Big[\left(\mathbf{V}^{(j)}\right)^{\frac{2(r-1)}{q}}\Big]_{k+n,k+n}\Big)^q\Big)\times$$

$$\times\mathbb{E}^{\frac{q-1}{q}}\Big|\Big[\mathbf{V}^{2(r-1)\frac{q}{q-1}}\Big]_{jj}\Big|^{\frac{q}{q-1}}.$$

Applying Hölder's inequality and interlacing theorem, we get

$$\Gamma_s^{(1)} \leqslant 2\sqrt{npq}\Big(\mathbb{E}Tr(\mathbf{V})^{2(r-1)}\Big) \leqslant$$

$$\leqslant 2\sqrt{npq}n^{\frac{1}{r}}\Big(\mathbb{E}Tr\mathbf{V}^{2r}\Big)^{\frac{r-1}{r}}, \tag{6}$$

$$\Gamma_s^{(2)} \leqslant Cq(np)^{1-2\varkappa-\frac{1}{q}}n^{\frac{1}{r}}\Big(\mathbb{E}Tr(\mathbf{V})^{2(r-1)}\Big) \leqslant$$

$$\leqslant Cq(np)^{1-2\varkappa-\frac{1}{q}}n^{\frac{1}{r}}\Big(\mathbb{E}Tr\mathbf{V}^{2r}\Big)^{\frac{r-1}{r}}. \tag{7}$$

Further, we consider $\widehat{A}_s$. Applying twice Hölder's inequality, we get

$$\widehat{A}_s \leqslant \Big(\sum_{j=1}^n \mathbb{E}\Big|\Big[\sum_{k=1}^m\sum_{l=1,l\neq k}^m X_{jk}\xi_{jk}X_{jl}\xi_{jl}\times$$

$$\times\Big[\left(\mathbf{V}^{(j)}\right)^{2s}\Big]_{k+n,l+n}\Big]\Big|^q\Big)^{\frac{1}{q}}\times$$

$$\times\Big(\sum_{j=1}^n \mathbb{E}\Big(\Big[\mathbf{V}^{2r-2s-2}\Big]_{jj}\Big)^{\frac{q}{q-1}}\Big)^{\frac{q-1}{q}}.$$

Applying inequality for quadratic forms (see [3]), we get

$$\widehat{A}_s \leqslant n^{\frac{1}{q}}\left(\widehat{\Gamma}_s^{(1)} + \widehat{\Gamma}_s^{(2)} + \widehat{\Gamma}_s^{(3)}\right)^{\frac{1}{q}}\times$$

$$\times\Big(\sum_{j=1}^n \mathbb{E}\Big(\Big[\mathbf{V}^{2r-2s-2}\Big]_{jj}\Big)^{\frac{q}{q-1}}\Big)^{\frac{q-1}{q}},$$

where

$$\widehat{\Gamma}_s^{(1)} = q^q p^q \mathbb{E}\Big(\sum_{k=1}^m\sum_{l=1}^m\Big(\Big[\mathbf{V}^{2s}\Big]_{k+n,l+n}\Big)^2\Big)^{\frac{q}{2}},$$

$$\widehat{\Gamma}_s^{(2)} = q^{\frac{3q}{2}}p^{\frac{q}{2}}(np)^{\frac{q}{2}-q\varkappa-2}p\times$$

$$\times\mathbb{E}\sum_{k=1}^m\Big(\sum_{l=1}^m\Big(\Big[\mathbf{V}^{2s}\Big]_{k+n,l+n}\Big)^2\Big)^{\frac{q}{2}},$$

$$\widehat{\Gamma}_s^{(3)} = q^{2q}(np)^{q-2\varkappa q-4}p^2\sum_{k=1}^m\sum_{l=1}^m \mathbb{E}\Big|\Big[\mathbf{V}^{2s}\Big]_{k+n,l+n}\Big|^q.$$

Note that

$$\left(\widehat{\Gamma}_s^{(1)}\right)^{\frac{1}{q}} \leqslant qp\mathbb{E}\Big(TrV^{4s}\Big)^{\frac{1}{2}} \leqslant$$

$$\leqslant qpn^{\frac{(r-2s)}{2r}}\mathbb{E}\Big(TrV^{2r}\Big)^{\frac{s}{r}}. \tag{8}$$

Further,

$$\mathbb{E}\sum_{k=1}^m\Big(\sum_{l=1}^m\Big(\Big[\mathbf{V}^{2s}\Big]_{k+n,l+n}\Big)^2\Big)^{\frac{q}{2}} \leqslant \mathbb{E}TrV^{2(r-1)}. \tag{9}$$

This implies that

$$\left(\widehat{\Gamma}_s^{(2)}\right)^{\frac{1}{q}} \leqslant q^{\frac{3}{2}}p^{\frac{1}{2}}(np)^{\frac{1}{2}-\varkappa-\frac{1}{q}}\frac{1}{n^{\frac{1}{q}}}\left(\mathbb{E}TrV^{2r}\right)^{\frac{s}{r}}. \tag{10}$$

Finally,

$$\left(\widehat{\Gamma}_s^{(3)}\right)^{\frac{1}{q}} \leqslant q^2(np)^{1-2\varkappa-\frac{2}{q}}\frac{1}{n^{\frac{2}{q}}}\Big(\mathbb{E}TrV^{2r}\Big)^{\frac{s}{r}}.$$

Moreover,

$$\Big(\sum_{j=1}^n \mathbb{E}\left(\left[\mathbf{V}^{2r-2s-2}\right]_{jj}\right)^{\frac{q}{q-1}}\Big)^{\frac{q-1}{q}} \leqslant$$

$$\leqslant n^{\frac{(q-1)}{qr}}\Big(Tr\mathbf{V}^{2r}\Big)^{\frac{r-s-1}{r}}.$$

Combining the estimates (8)–(10), we get

$$\widehat{A}_s \leqslant Cnp(\Sigma_1 + \Sigma_2 + \Sigma_3)\Big(\mathbb{E}Tr(\mathbf{V})^{2r}\Big)^{\frac{r-1}{r}},$$

where

$$\Sigma_1 = qn^{-\frac{1}{2}+\frac{1}{r}},$$

$$\Sigma_2 = q^{\frac{3}{2}}(np)^{-\frac{1}{2}-(\varkappa+\frac{1}{q})}n^{-\frac{1}{2}+\frac{1}{r}(1-\frac{s}{r-1})},$$

$$\Sigma_3 = \frac{q^2}{(np)^{2\varkappa+\frac{2}{q}}}n^{\frac{1}{r}-\frac{2s}{r(r-1)}}.$$

Summing by $s = 1, \ldots, [\frac{r-1}{2}]$, we get

$$\sum_{s=0}^{[\frac{r-1}{2}]}\widehat{A}_s \leqslant Cnpn^{\frac{1}{r}}\Big(\frac{r\log r}{\sqrt{n}} + \frac{r^{\frac{3}{2}}}{(np)^{\frac{1}{2}+\varkappa}\sqrt{n}} + \frac{r^2}{(np)^{2\varkappa}}\Big)\times$$

$$\times\Big(\mathbb{E}Tr\mathbf{V}^{2r}\Big)^{\frac{r-1}{r}}.$$

Now we estimate B_s. First we note that $B_0 = 0$. We can assume that $s \geqslant 1$. We represent B_s in the form

$$B_s = \sum_{j=1}^n \mathbb{E}\Big[\sum_{k=1}^m\sum_{l=1,l\neq j} X_{jl}X_{jk}\xi_{jl}\xi_{jk}\times$$

$$\times\Big[\left(\mathbf{V}^{(j)}\right)^{2s}\Big]_{l+n,k+n}\Big]\left(\left[\mathbf{V}^{2r-2s-2}\right]_{jj}\right).$$

The estimation of B_s is similar to the estimation of $\widehat{A}_s$. We get

$$\sum_{s=0}^{[\frac{r-1}{2}]}|B_s| \leqslant Cnpn^{\frac{1}{r}}\Bigg(\frac{r\log r}{\sqrt{n}} + \frac{r^{\frac{3}{2}}}{(np)^{\frac{1}{2}+\varkappa}\sqrt{n}} +$$

$$+\frac{r^2}{(np)^{2\varkappa}}\Big)\Big(\mathbb{E}Tr(\mathbf{V})^{2r}\Big)^{\frac{r-1}{r}}.$$

In conclusion, we estimate Z_r. Without loss of generality, we can assume that r is even. First we write

$$|Z_r|\leqslant\sum_{j=1}^{n}\sum_{l=1}^{m}\mathbb{E}\Big|\sum_{k=1}^{m}X_{jk}\xi_{jk}\left[\left(\mathbf{V}^{(j)}\right)^{r-1}\right]_{k+n,l}\Big|\times$$
$$\times\Big|\,[\mathbf{V}^r]_{l,j}\,\Big|.$$

Applying Cauchy inequality, we obtain

$$|Z_r|\leqslant\sum_{j=1}^{n}\sum_{l=1}^{m}\mathbb{E}^{\frac12}\left|\sum_{k=1}^{m}X_{jk}\xi_{jk}\left[\left(\mathbf{V}^{(j)}\right)^{r-1}\right]_{k+n,l}\right|^2\times$$
$$\times\mathbb{E}^{\frac12}\big|[\mathbf{V}^r]_{l,j}\big|^2$$

Taking the conditional expectation and applying Cauchy inequality again, we get

$$|Z_r|\leqslant\sum_{j=1}^{n}\Big(p\mathbb{E}\sum_{l=1}^{m}\sum_{k=1}^{m}\Big|\left[\left(\mathbf{V}^{(j)}\right)^{r-1}\right]_{k+n,l}\Big|^2\Big)^{\frac12}\times$$
$$\times\Big(\mathbb{E}\sum_{l=1}^{m}|\,[\mathbf{V}^r]_{l,j}\,|^2\Big)^{\frac12}.$$

From here it follows that

$$|Z_r|\leqslant C\sqrt{p}\sum_{j=1}^{n}\Big(\mathbb{E}Tr\left[\mathbf{V}^{(j)}\right]^{2r-2}\Big)^{\frac12}\times$$
$$\times\Big(\mathbb{E}\sum_{l=1}^{m}|\,[\mathbf{V}^r]_{l,j}\,|^2\Big)^{\frac12}.$$

Taking into account interlacing theorem and applying Cauchy inequality, we get

$$|Z_r|\leqslant C\sqrt{np}\mathbb{E}^{\frac12}Tr[\mathbf{V}]^{2r-2}\Big(\mathbb{E}\sum_{j=1}^{n}\sum_{l=1}^{m}|[\mathbf{V}^r]_{l,j}|^2\Big)^{\frac12}\leqslant$$

$$\leqslant Cn^{\frac{1}{2r}}\sqrt{np}\mathbb{E}^{\frac{2r-1}{2r}}Tr\mathbf{V}^{2r}. \qquad (11)$$

Combining inequalities (6), (7), (11), and estimates of corresponding sums, we get the required. ☐

Proof of Theorem 1. The proof of Theorem 1 follows now from Corollary 2 by choosing t sufficiently large. ☐

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