

УДК 521.1
DOI 10.19110/1994-5655-2021-6-20-35

С.М. ПОЛЕЩИКОВ

**ПОТЕНЦИАЛЫ, ДОПУСКАЮЩИЕ
ИНТЕГРИРОВАНИЕ ВОЗМУЩЕННОЙ
ЗАДАЧИ ДВУХ ТЕЛ В РЕГУЛЯРНЫХ
КООРДИНАТАХ**

г. Сыктывкар

polism@list.ru

S.M. POLESHCHIKOV

**POTENTIALS ALLOWING INTEGRATION OF
THE PERTURBED TWO-BODY PROBLEM IN
REGULAR COORDINATES**

Syktывkar

Аннотация

Изучается задача разделения переменных в некоторых системах координат, полученных с помощью L -преобразований. Даны потенциалы, допускающие разделение регулярных переменных в возмущенной задаче двух тел. Потенциалы содержат произвольные гладкие функции. Рассмотрен пример потенциала, приводящий к построению явного решения задачи в эллиптических функциях. Выделены случаи ограниченного и неограниченного движения. Приведены результаты численных экспериментов.

Ключевые слова:

возмущенная задача двух тел, L -матрицы, интегрируемость, эллиптические функции

Abstract

The problem of separation of variables in some coordinate systems obtained with the use of L -transformations is studied. Potentials are shown that allow separation of regular variables in a perturbed two-body problem. The potential contains two arbitrary smooth functions. An example of a potential is considered allowing explicit solution of the problem in terms of elliptic functions. The cases of bounded and unbounded motion are shown. The results of numerical experiments are given.

Keywords:

perturbed two-body problem, L -matrices, integrability, elliptic functions

Introduction

The integrable cases of motion equations have great practical value. Their significance is determined by the fact that with the help of their solutions one can analyze the motion. In a number of cases integrable problems are used to construct intermediate orbits [1,2]. One of non-trivial examples of integrated systems is the particle motion in a Newtonian field with additional constant acceleration vector. This had been investigated earlier by a number of authors [3–5] and applied to analysis of space flights with constant jet acceleration. In 1970, this problem was studied using regular coordinates obtained from the KS-matrix [6]. In contrast to [6], in [7] integration of the same problem was performed in regular coordinates obtained with the use of L -transformations.

In the present work we consider a problem of constructing potentials allowing integration of the equations of motion. The idea of our approach consists in the following. First, a new dynamic system is constructed, having more degrees of freedom than the original one. To do this, an L -transformation is applied. The theory of L -matrices and their applications is given in [8,9]. Using new coordinates, a general potential is selected, allowing separation of variables in the Hamilton - Jacobi equation. After this, an inverse transform to original coordinates is performed, using explicit formulas. As a basis for selecting general potential with the required integrability property, a well known Stackel theorem is used [10]. This theorem gives necessary and sufficient conditions for separation of variables for orthogonal Hamilton systems, i.e. systems whose Hamiltonian contains only squares of generalized momentums.

Note that separation of variables depends on a choice of a coordinate system. We consider here three

kinds of coordinate systems: regular, bipolar and spherical. The last two systems are introduced in regular coordinates. Canonical equations in regular coordinates are constructed using arbitrary L -transformations from the initial canonical motion equations of the perturbed two-body problem. The new equations have also orthogonal form and are invariant with respect to L -similarity transforms. In the nonperturbed case these equations do not have singularity at the attracting center. Due to invariance with respect to some perturbing potentials allowing integrability, one can introduce two additional angular parameters.

As a result of this approach the general solution of original system is represented in parametric form, where fictitious time plays the role of parameter, while the physical time depends on this fictitious time and initial data. This sort of integrability is sometimes called 'Sundman integrability' [11].

As an example of integrable case of the perturbed two-body problem the special kind of potential is given. In this example the explicit solution of the problem in terms of elliptic functions is expressed, and the criterion of bounded motion is formulated.

Notation. Everywhere below vectors are regarded as column vectors, and are given in bold letters. The sign T placed over the vector or matrix symbol denotes transposition. A quantity evaluated at the initial moment of physical or fictitious time is denoted by zero superscript: $f(0) \equiv f^0$.

1. The separation of variables

Let us consider the Hamiltonian function of the perturbed two-body problem

$$H = H(\mathbf{x}, \mathbf{y}) = \frac{1}{2}|\mathbf{y}|^2 - \frac{\mu}{r} + V, \quad (1)$$

$$\mu = \gamma(m + m_0), \quad r = |\mathbf{x}|,$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$ is the position vector of the point of mass m with respect to the point of mass m_0 ; $\mathbf{y} = (y_1, y_2, y_3)^T$ is the generalized impulses ($y_i = \dot{x}_i, i = 1, 2, 3$); γ is the gravitational constant; $V = V(\mathbf{x})$ is the perturbed potential.

For construction of the equations of motion in regular coordinates we will need the L -transformation $\mathbf{z} = L(\mathbf{q})\mathbf{q}$ generated by the L -matrix of the fourth order that has the following properties:

$$L(\mathbf{q})L^T(\mathbf{q}) = L^T(\mathbf{q})L(\mathbf{q}) = |\mathbf{q}|^2 E \quad \forall \mathbf{q} \in \mathbf{R}^4, \quad (2)$$

$$(L(\mathbf{q})\mathbf{p})_i = (L(\mathbf{p})\mathbf{q})_i, \quad i = 1, \dots, p, \quad (3)$$

$$(L(\mathbf{q})\mathbf{p})_i = -(L(\mathbf{p})\mathbf{q})_i, \quad i = p + 1, \dots, 4 \quad (4)$$

$$\forall \mathbf{q}, \mathbf{p} \in \mathbf{R}^4.$$

Here E is the unitary matrix. The conditions (2) – (4) simultaneously hold only for $p = 1$ or $p = 3$. The quantity p is the rank of L -transformation. The following theorem can be proved [8, 9].

Theorem 1. *An arbitrary L -matrix generating L -transformation of rank three, has the form*

$$L(\mathbf{q}) = \begin{pmatrix} \mathbf{q}^T K_1 K_4 \\ \mathbf{q}^T K_2 K_4 \\ \mathbf{q}^T K_3 K_4 \\ \mathbf{q}^T K_4 \end{pmatrix}, \quad (5)$$

where orthogonal skew-symmetric matrices K_1, K_2, K_3, K_4 are equal to either

$$\begin{aligned} K_i &= a_{1i}\mathcal{U} + a_{2i}\mathcal{V} + a_{3i}\mathcal{W}, \quad i = 1, 2, 3, \\ K_4 &= a_1\mathcal{X} + a_2\mathcal{Y} + a_3\mathcal{Z}, \end{aligned} \quad (6)$$

or

$$\begin{aligned} K_i &= a_{1i}\mathcal{X} + a_{2i}\mathcal{Y} + a_{3i}\mathcal{Z}, \quad i = 1, 2, 3, \\ K_4 &= a_1\mathcal{U} + a_2\mathcal{V} + a_3\mathcal{W}. \end{aligned} \quad (7)$$

The triplet of vectors $\mathbf{e}_i = (a_{1i}, a_{2i}, a_{3i})^T, i = 1, 2, 3$, forms an orthonormal basis in \mathbf{R}^3 , and $\mathbf{e} = (a_1, a_2, a_3)^T$ is an arbitrary unitary vector.

Conversely, the arbitrary four skew-symmetric matrices in the form (6) or (7) define the L -matrix by the formula (5).

In the formulae (6) and (7) there are the so-called basic skew-symmetric orthogonal matrices

$$\mathcal{U} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{W} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{Y} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{Z} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices K_i are called *generators* of the L -matrix. If K_1, K_2, K_3, K_4 are calculated by the formulae (6) then $L(\mathbf{q})$ is called *the L -matrix of the first type*, otherwise *the L -matrix of the second type*.

We transfer from variables t, x_i, y_i to the new variables τ, q_j, p_j by the formulae

$$dt = r d\tau,$$

$$\begin{cases} \mathbf{x} = \Lambda(\mathbf{q})\mathbf{q}, \\ \mathbf{y} = \frac{1}{2|\mathbf{q}|^2}\Lambda(\mathbf{q})\mathbf{p}, \quad \mathbf{q}, \mathbf{p} \in \mathbf{R}^4 \end{cases} \quad (8)$$

where the matrix $\Lambda(\mathbf{q})$ is found from (5) by rejection of the fourth line:

$$\Lambda(\mathbf{q}) = \begin{pmatrix} \mathbf{q}^T K_1 K_4 \\ \mathbf{q}^T K_2 K_4 \\ \mathbf{q}^T K_3 K_4 \end{pmatrix}.$$

Consider the equations of motion in new variables q_i, p_i

$$\frac{dq_j}{d\tau} = \frac{\partial \mathcal{K}}{\partial p_j}, \quad \frac{dp_j}{d\tau} = -\frac{\partial \mathcal{K}}{\partial q_j}, \quad j = 0, 1, 2, 3, 4 \quad (9)$$

with the Hamiltonian

$$\begin{aligned} \mathcal{K} &= \frac{1}{8}|\mathbf{p}|^2 + p_0|\mathbf{q}|^2 + |\mathbf{q}|^2 V_c(\mathbf{q}), \\ V_c(\mathbf{q}) &= V(\mathbf{x}(\mathbf{q})). \end{aligned} \quad (10)$$

In this system the first equation with $j = 0$ corresponds to transformation of time: $dq_0 = |\mathbf{q}|^2 d\tau$. The variable p_0 is conjugate to q_0 and has a constant value.

If
$$\mathbf{x}(0) = \mathbf{x}^0, \quad \mathbf{y}(0) = \mathbf{y}^0 \quad (11)$$

are initial conditions for the variables of the system with the Hamiltonian (1), then, as it is proved in [12, 13], with the initial values defined by formulae

$$\begin{cases} q_0(0) = 0, & \mathbf{x}^0 = \Lambda(\mathbf{q}^0)\mathbf{q}^0, \\ p_0(0) = -H(\mathbf{x}^0, \mathbf{y}^0), & \mathbf{p}^0 = 2\Lambda^T(\mathbf{q}^0)\mathbf{y}^0, \end{cases} \quad (12)$$

the solution of (9) becomes, under the transformation (8), a solution of the system with the Hamiltonian (1) satisfying the initial conditions (11). The function $\mathbf{q}^T K_4 \mathbf{p}$ preserves a constant value along solutions of (9), and with the initial conditions from (12), this value is zero [13]. Hence, the equality $\mathbf{q}^T K_4 \mathbf{p} = 0$ is the first integral of this system. The variable q_0 coincides with physical time t .

Note that the systems with Hamiltonian (1) and (10) have different orders. The choice of initial values by the formulae (12) means that there is a special construction of the system (9) for each trajectory of the system with Hamiltonian (1).

Let's pick up the form of potential V , admitting division of variables. For this purpose we shall take advantage of the theorem proved by Stackel [10].

Theorem 2. *The system with Hamiltonian*

$$H = \sum_{i=1}^n c_i(q_1, \dots, q_n) \left(\frac{1}{2} p_i^2 + V_i(q_i) \right),$$

admits separation of variables in the Hamilton - Jacobi equation if and only if there is a nonspecial matrix Φ of order n which elements φ_{si} depend only on q_i , such as

$$\Phi \mathbf{c} = (1, 0, \dots, 0)^T, \quad (13)$$

where $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$.

In this case the integrals of motion will be

$$\begin{aligned} t - \beta_1 &= \sum_{i=1}^n \int_{q_i^0}^{q_i} \frac{\varphi_{1i}(q_i) dq_i}{\sqrt{f_i(q_i)}}, \\ -\beta_s &= \sum_{i=1}^n \int_{q_i^0}^{q_i} \frac{\varphi_{si}(q_i) dq_i}{\sqrt{f_i(q_i)}}, \quad s = 2, \dots, n, \\ p_i &= \sqrt{f_i(q_i)}, \quad i = 1, \dots, n, \end{aligned} \quad (14)$$

where $f_i(q_i) = 2(\alpha_1 \varphi_{1i}(q_i) + \dots + \alpha_n \varphi_{ni}(q_i) - V_i(q_i))$; α_i, β_i ($i = 1, \dots, n$) is constant. As q_i^0 a simple root of the function $f_i(q_i)$ is taken.

Consider again the separation of variables in regular coordinates q_i . The Hamiltonian looks like (10). In this case we have

$$\begin{aligned} c_1 = c_2 = c_3 = c_4 &= \frac{1}{4}, \\ |\mathbf{q}|^2(p_0 + V_c(\mathbf{q})) &= \frac{1}{4} \sum_{s=1}^4 V_s(q_s). \end{aligned}$$

The solution of system (13) will be, for example, the matrix

$$\Phi = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The potential is defined up to a constant. As p_0 is a constant, we obtain

$$\begin{aligned} V_c(\mathbf{q}) &= \frac{1}{4|\mathbf{q}|^2} (V_1(q_1) + V_2(q_2) + \\ &+ V_3(q_3) + V_4(q_4)). \end{aligned} \quad (15)$$

Let's find expression for the potential $V_c(\mathbf{q})$ in original coordinates \mathbf{x} . We notice that variables x_i and r are quadratic forms of the variables q_1, q_2, q_3, q_4 . Using the L -similarity transformation it is possible to choose an L -matrix such as a linear combination $B_1 x_1 + B_2 x_2 + B_3 x_3$ that will be equal to the sum of squares of q_i with some coefficients. Note that for any L -matrix we have $r = |\mathbf{q}|^2$. As $V_c(\mathbf{q})$ is to be of the form (15), the required potential in x -coordinates will be the function of the form

$$V(\mathbf{x}) = \frac{1}{r} (Ar + B_1 x_1 + B_2 x_2 + B_3 x_3). \quad (16)$$

Let's specify a choice of L -matrix with the required property. Introduce the notation

$$B = \sqrt{B_1^2 + B_2^2 + B_3^2}, \quad b_i = \frac{B_i}{B}, \quad i = 1, 2, 3.$$

Suppose that the L -matrix is of the first type. That is, K_1, K_2, K_3 are calculated by the formula (6); for simplicity we assume that $K_4 = -\mathcal{Y}$. Then

$$\begin{aligned} Ar + B(b_1 x_1 + b_2 x_2 + b_3 x_3) &= \\ &= Ar - B \mathbf{q}^T \left[(b_1 a_{11} + b_2 a_{12} + b_3 a_{13}) \mathcal{U} + \right. \\ &+ (b_1 a_{21} + b_2 a_{22} + b_3 a_{23}) \mathcal{V} + \\ &+ (b_1 a_{31} + b_2 a_{32} + b_3 a_{33}) \mathcal{W} \left. \right] \mathcal{Y} \mathbf{q}. \end{aligned}$$

Choose the parameters a_{ij} of L -matrix in such a way that the following equalities hold:

$$\begin{cases} b_1 a_{11} + b_2 a_{12} + b_3 a_{13} = 1, \\ b_1 a_{21} + b_2 a_{22} + b_3 a_{23} = 0, \\ b_1 a_{31} + b_2 a_{32} + b_3 a_{33} = 0. \end{cases} \quad (17)$$

Geometrically, the solution to this system means that the vector $\mathbf{i}_1 = (a_{11}, a_{12}, a_{13})^T$ coincides with $\mathbf{b} = (b_1, b_2, b_3)^T$, and the vectors $\mathbf{i}_2 = (a_{21}, a_{22}, a_{23})^T$, $\mathbf{i}_3 = (a_{31}, a_{32}, a_{33})^T$ are orthogonal to \mathbf{b} . Moreover, it follows from the structure of the L -matrix that vectors $\mathbf{i}_1, \mathbf{i}_2$, and \mathbf{i}_3 form a frame. It is evident that the system (17) has infinite number of solutions. We write its general solution. For the first vector we have

$$\mathbf{i}_1 = (b_1, b_2, b_3)^T.$$

For \mathbf{i}_2 and \mathbf{i}_3 we assume, in the case $b_1^2 + b_2^2 \neq 0$, that

$$\mathbf{i}_2 = \frac{1}{\sqrt{b_1^2 + b_2^2}} \left(b_2 \cos \alpha + b_1 b_3 \sin \alpha, -b_1 \cos \alpha + b_2 b_3 \sin \alpha, -(b_1^2 + b_2^2) \sin \alpha \right)^T,$$

$$\mathbf{i}_3 = \frac{1}{\sqrt{b_1^2 + b_2^2}} \left(-b_2 \sin \alpha + b_1 b_3 \cos \alpha, b_1 \sin \alpha + b_2 b_3 \cos \alpha, -(b_1^2 + b_2^2) \cos \alpha \right)^T.$$

If $b_1^2 + b_2^2 = 0$, then $\mathbf{b} = (0, 0, b_3)^T$, $b_3 = \pm 1$. Therefore, we can take the following vectors as the general solution of the system (17):

$$\mathbf{i}_1 = (0, 0, b_3)^T, \quad \mathbf{i}_2 = (\cos \alpha, \sin \alpha, 0)^T, \\ \mathbf{i}_3 = b_3 (-\sin \alpha, \cos \alpha, 0)^T.$$

The quantity $\alpha \in [0, 2\pi]$ plays the role of an arbitrary parameter of the general solution.

After choosing the parameters a_{ij} , the matrix $\Lambda(\mathbf{q})$ is determined uniquely. The solution of (17) gives

$$V_c(\mathbf{q}) = \frac{1}{|\mathbf{q}|^2} (A|\mathbf{q}|^2 - B\mathbf{q}^T \mathcal{U} \mathcal{Y} \mathbf{q}) = \\ = \frac{1}{|\mathbf{q}|^2} ((A+B)q_1^2 + (A+B)q_2^2 + (A-B)q_3^2 + (A-B)q_4^2).$$

Hamiltonian in q -coordinates corresponding to this potential becomes

$$\mathcal{K} = \sum_{i=1}^4 \frac{1}{4} \left(\frac{1}{2} p_i^2 + 4p_0 q_i^2 + 4D_i q_i^2 \right),$$

where $D_1 = D_2 = A + B$, $D_3 = D_4 = A - B$. The canonical system of the equations falls into four subsystems

$$\frac{dq_i}{d\tau} = \frac{1}{4} p_i, \quad \frac{dp_i}{d\tau} = -2(p_0 + D_i) q_i, \quad i = 1, 2, 3, 4. \quad (18)$$

These systems are equivalent to four harmonious oscillators. Integrals of motion are obtained either from (14), or directly from solving (18). Thus, separation of variables for potential (16) is carried out.

For regular q -coordinates, we introduce a new coordinate system. To preserve the canonical form of equations of motion, we use the canonical transformation with generating function

$$\Psi = p_1 \sqrt{Q_1} \cos Q_2 + p_2 \sqrt{Q_1} \sin Q_2 + \\ + p_3 \sqrt{Q_3} \cos Q_4 + p_4 \sqrt{Q_3} \sin Q_4.$$

We obtain

$$q_1 = \frac{\partial \Psi}{\partial p_1} = \sqrt{Q_1} \cos Q_2, \\ q_2 = \frac{\partial \Psi}{\partial p_2} = \sqrt{Q_1} \sin Q_2, \\ q_3 = \frac{\partial \Psi}{\partial p_3} = \sqrt{Q_3} \cos Q_4, \\ q_4 = \frac{\partial \Psi}{\partial p_4} = \sqrt{Q_3} \sin Q_4, \\ P_1 = \frac{\partial \Psi}{\partial Q_1} = \frac{1}{2\sqrt{Q_1}} (p_1 \cos Q_2 + p_2 \sin Q_2), \\ P_2 = \frac{\partial \Psi}{\partial Q_2} = \sqrt{Q_1} (-p_1 \sin Q_2 + p_2 \cos Q_2), \\ P_3 = \frac{\partial \Psi}{\partial Q_3} = \frac{1}{2\sqrt{Q_3}} (p_3 \cos Q_4 + p_4 \sin Q_4), \\ P_4 = \frac{\partial \Psi}{\partial Q_4} = \sqrt{Q_3} (-p_3 \sin Q_4 + p_4 \cos Q_4). \quad (19)$$

The coordinates Q_1, Q_2, Q_3, Q_4 , obtained from (19), will be called bipolar. From the last four equations we find p_1, p_2, p_3, p_4 :

$$p_1 = 2P_1 \sqrt{Q_1} \cos Q_2 - \frac{P_2}{\sqrt{Q_1}} \sin Q_2, \\ p_2 = 2P_1 \sqrt{Q_1} \sin Q_2 + \frac{P_2}{\sqrt{Q_1}} \cos Q_2, \\ p_3 = 2P_3 \sqrt{Q_3} \cos Q_4 - \frac{P_4}{\sqrt{Q_3}} \sin Q_4, \\ p_4 = 2P_3 \sqrt{Q_3} \sin Q_4 + \frac{P_4}{\sqrt{Q_3}} \cos Q_4. \quad (20)$$

In the new variables the Hamiltonian \mathcal{K} becomes

$$\bar{\mathcal{K}} = \frac{1}{8} \left(4Q_1 P_1^2 + \frac{P_2^2}{Q_1} + 4Q_3 P_3^2 + \frac{P_4^2}{Q_3} \right) + \\ + p_0 (Q_1 + Q_3) + (Q_1 + Q_3) \bar{V},$$

where function \bar{V} is expressed in terms of Q_i .

Similar to the above, consider separation of variables in bipolar coordinates. In the notations of theorem 2 we now have

$$c_1 = Q_1, \quad c_2 = \frac{1}{4Q_1}, \quad c_3 = Q_3, \quad c_4 = \frac{1}{4Q_3}.$$

As a solution to (13) one can take the matrix

$$\Phi = \begin{pmatrix} \frac{1}{Q_1} & 0 & 0 & 0 \\ -\frac{1}{4Q_1^2} & 1 & 0 & 0 \\ -\frac{1}{Q_1} & 0 & \frac{1}{Q_3} & 0 \\ 0 & 0 & -\frac{1}{4Q_3^2} & 1 \end{pmatrix}. \quad (21)$$

For the potential \bar{V} admitting separation of variables, we

find

$$\bar{V} = \frac{1}{Q_1 + Q_3} \left(Q_1 \bar{V}_1(Q_1) + \frac{1}{4Q_1} \bar{V}_2(Q_2) + Q_3 \bar{V}_3(Q_3) + \frac{1}{4Q_3} \bar{V}_4(Q_4) \right).$$

In q -coordinates we obtain the form

$$V_c = \frac{1}{|\mathbf{q}|^2} \left((q_1^2 + q_2^2) \bar{V}_1(q_1^2 + q_2^2) + \frac{\bar{V}_2(\arctan \frac{q_2}{q_1})}{4(q_1^2 + q_2^2)} + (q_3^2 + q_4^2) \bar{V}_3(q_3^2 + q_4^2) + \frac{\bar{V}_4(\arctan \frac{q_4}{q_3})}{4(q_3^2 + q_4^2)} \right).$$

Passing to x -coordinates, we use the concrete L -transformation

$$\begin{cases} x_1 = 2q_1q_4 + 2q_2q_3, \\ x_2 = -2q_1q_3 + 2q_2q_4, \\ x_3 = q_1^2 + q_2^2 - q_3^2 - q_4^2, \end{cases} \quad (22)$$

which follows from (5), (6) with $K_1 = \mathcal{V}$, $K_2 = \mathcal{W}$, $K_3 = \mathcal{U}$, $K_4 = -\mathcal{Y}$. Taking into account that for any L -matrix the equality $r = q_1^2 + q_2^2 + q_3^2 + q_4^2$ holds, we obtain

$$q_3^2 + q_4^2 = \frac{1}{2}(r - x_3), \quad q_1^2 + q_2^2 = \frac{1}{2}(r + x_3).$$

The general solution of the first equation is

$$q_3 = \sqrt{\frac{r - x_3}{2}} \cos \psi, \quad q_4 = \sqrt{\frac{r - x_3}{2}} \sin \psi, \\ \psi \in [0, 2\pi].$$

Then

$$q_1 = \frac{x_1 \sin \psi - x_2 \cos \psi}{\sqrt{2}\sqrt{r - x_3}}, \quad q_2 = \frac{x_1 \cos \psi + x_2 \sin \psi}{\sqrt{2}\sqrt{r - x_3}}.$$

In a similar way we may introduce a parameter, using the second equation,

$$q_1 = \sqrt{\frac{r + x_3}{2}} \cos \psi_1, \quad q_2 = \sqrt{\frac{r + x_3}{2}} \sin \psi_1, \\ \psi_1 \in [0, 2\pi].$$

As is well known [12], with L -transformation for a point in \mathbf{R}^3 at a distance r from the origin, there corresponds a point of some circle of radius \sqrt{r} in \mathbf{R}^4 . The variables q_i contain an arbitrary parameter ψ (or ψ_1), giving parametrization of the given circle. In the original coordinates x_i this parameter disappears. Note that

$$\frac{q_2}{q_1} = \tan \psi_1, \quad \frac{q_4}{q_3} = \tan \psi.$$

We therefore assume functions V_2, V_4 to be constant. Then we arrive at a potential of the form

$$V(\mathbf{x}) = \frac{1}{r} \left[G_1((r + x_3)/2) + G_2((r - x_3)/2) \right], \quad (23)$$

where G_1, G_2 are arbitrary smooth functions. The Hamiltonian in bipolar coordinates for this potential takes

the form

$$\bar{K} = Q_1 \left(\frac{P_1^2}{2} + p_0 + \frac{G_1(Q_1)}{Q_1} \right) + \frac{1}{4Q_1} \frac{P_2^2}{2} + Q_3 \left(\frac{P_3^2}{2} + p_0 + \frac{G_2(Q_3)}{Q_3} \right) + \frac{1}{4Q_3} \frac{P_4^2}{2}.$$

In view of the solution (21) for f_i from the theorem 2 we have

$$f_1(Q_1) = 2 \left(\frac{\alpha_1}{Q_1} - \frac{\alpha_2}{4Q_1^2} - \frac{\alpha_3}{Q_1} - p_0 - \frac{G_1(Q_1)}{Q_1} \right), \\ f_2(Q_2) = 2\alpha_2,$$

$$f_3(Q_3) = 2 \left(\frac{\alpha_3}{Q_3} - \frac{\alpha_4}{4Q_3^2} - p_0 - \frac{G_2(Q_3)}{Q_3} \right), \\ f_4(Q_4) = 2\alpha_4.$$

Then integrals of motion are obtained by formulas (14).

Let's consider one more case of separation of variables. Introduce the spherical coordinates in q -coordinates

$$\begin{aligned} q_1 &= \sqrt{Q_1} \cos Q_2 \cos Q_4, \\ q_2 &= \sqrt{Q_1} \sin Q_2 \cos Q_4, \\ q_3 &= \sqrt{Q_1} \cos Q_3 \sin Q_4, \\ q_4 &= \sqrt{Q_1} \sin Q_3 \sin Q_4. \end{aligned} \quad (24)$$

We supplement the transformation (24) to obtain a canonical transformation of impulses

$$\begin{aligned} p_1 &= 2\sqrt{Q_1} \cos Q_2 \cos Q_4 P_1 - \\ &\quad - \frac{\sin Q_2}{\sqrt{Q_1} \cos Q_4} P_2 - \frac{\cos Q_2 \sin Q_4}{\sqrt{Q_1}} P_4, \\ p_2 &= 2\sqrt{Q_1} \sin Q_2 \cos Q_4 P_1 + \\ &\quad + \frac{\cos Q_2}{\sqrt{Q_1} \cos Q_4} P_2 - \frac{\sin Q_2 \sin Q_4}{\sqrt{Q_1}} P_4, \\ p_3 &= 2\sqrt{Q_1} \cos Q_3 \sin Q_4 P_1 - \\ &\quad - \frac{\sin Q_3}{\sqrt{Q_1} \sin Q_4} P_3 + \frac{\cos Q_3 \cos Q_4}{\sqrt{Q_1}} P_4, \\ p_4 &= 2\sqrt{Q_1} \sin Q_3 \sin Q_4 P_1 + \\ &\quad + \frac{\cos Q_3}{\sqrt{Q_1} \sin Q_4} P_3 + \frac{\sin Q_3 \cos Q_4}{\sqrt{Q_1}} P_4. \end{aligned} \quad (25)$$

Then in new variables the Hamiltonian will be

$$\bar{K} = \frac{1}{8} \left(4Q_1 P_1^2 + \frac{P_2^2}{Q_1 \cos^2 Q_4} + \frac{P_3^2}{Q_1 \sin^2 Q_4} + \frac{P_4^2}{Q_1} \right) + p_0 Q_1 + Q_1 \bar{V}.$$

In the notations of Stackel theorem we have

$$\begin{aligned} c_1 &= Q_1, \quad c_2 = \frac{1}{4Q_1 \cos^2 Q_4}, \\ c_3 &= \frac{1}{4Q_1 \sin^2 Q_4}, \quad c_4 = \frac{1}{4Q_1}. \end{aligned}$$

In this case the solution of (13) will be the matrix

$$\Phi = \begin{pmatrix} \frac{1}{Q_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{\cos^2 Q_4} \\ 0 & 0 & 1 & -\frac{1}{\sin^2 Q_4} \\ \frac{1}{4Q_1^2} & 0 & 0 & -1 \end{pmatrix}. \quad (26)$$

The potential \bar{V} , admitting separation of variables, can be written as

$$\bar{V} = \frac{1}{Q_1} \left(Q_1 \bar{V}_1(Q_1) + \frac{1}{4Q_1 \cos^2 Q_4} \bar{V}_2(Q_2) + \frac{1}{4Q_1 \sin^2 Q_4} \bar{V}_3(Q_3) + \frac{1}{4Q_1} \bar{V}_4(Q_4) \right).$$

In view of relations

$$Q_1 \cos^2 Q_4 = q_1^2 + q_2^2 = \frac{r + x_3}{2},$$

$$Q_1 \sin^2 Q_4 = q_3^2 + q_4^2 = \frac{r - x_3}{2},$$

$$Q_1 = |\mathbf{q}|^2 = r,$$

$$\tan^2 Q_4 = \frac{q_3^2 + q_4^2}{q_1^2 + q_2^2} = \frac{1 - x_3/r}{1 + x_3/r},$$

$$Q_2 = \arctan \frac{q_2}{q_1}, \quad Q_3 = \arctan \frac{q_4}{q_3}$$

following from (22), (24), and the remarks above, we obtain the required form of potential in x -coordinates

$$V(\mathbf{x}) = \frac{1}{r} \left[G_1(r) + \frac{2A}{r + x_3} + \frac{2B}{r - x_3} + \frac{1}{r} G_2\left(\frac{x_3}{r}\right) \right], \quad (27)$$

where G_1, G_2 are arbitrary smooth functions and A, B arbitrary constants.

Now assume that a Hamiltonian (1) with the potential (27) is given. Applying L -transformation (22), we write the new Hamiltonian in q -coordinates as

$$\begin{aligned} \mathcal{K} = & \frac{1}{8} |\mathbf{p}|^2 + p_0 |\mathbf{q}|^2 + G_1(|\mathbf{q}|^2) + \\ & + \frac{A}{q_1^2 + q_2^2} + \frac{B}{q_3^2 + q_4^2} + \\ & + \frac{1}{|\mathbf{q}|^2} G_2\left(\frac{q_1^2 + q_2^2 - q_3^2 - q_4^2}{|\mathbf{q}|^2}\right). \end{aligned}$$

Fulfilling canonical transformation (24), (25), we have

$$\begin{aligned} \bar{\mathcal{K}} = & Q_1 \left(\frac{P_1^2}{2} + p_0 + \frac{G_1(Q_1)}{Q_1} \right) + \\ & + \frac{1}{4Q_1 \cos^2 Q_4} \left(\frac{P_2^2}{2} + 4A \right) + \\ & + \frac{1}{4Q_1 \sin^2 Q_4} \left(\frac{P_3^2}{2} + 4B \right) + \\ & + \frac{1}{4Q_1} \left(\frac{P_4^2}{2} + 4G_2(\cos 2Q_4) \right). \end{aligned}$$

Taking into consideration matrix (26), we then obtain

$$\begin{aligned} f_1(Q_1) &= 2 \left(\frac{\alpha_1}{Q_1} + \frac{\alpha_4}{4Q_1^2} - p_0 - \frac{G_1(Q_1)}{Q_1} \right), \\ f_2(Q_2) &= 2(\alpha_2 - 4A), \quad f_3(Q_3) = 2(\alpha_3 - 4B), \\ f_4(Q_4) &= 2 \left(-\frac{\alpha_2}{\cos^2 Q_4} - \frac{\alpha_3}{\sin^2 Q_4} - \right. \\ & \left. - \alpha_4 - 4G_2(\cos 2Q_4) \right). \end{aligned}$$

The integrals of motion follow from (14).

Note that using an arbitrary L -transformations allows one to introduce two parameters into the potentials obtained. These two parameters are determined by some constant unit vector \mathbf{b} . For example, instead of (27) one can write

$$V(\mathbf{x}) = \frac{1}{r} \left[G_1(r) + \frac{2A}{r + \mathbf{b}^T \mathbf{x}} + \frac{2B}{r - \mathbf{b}^T \mathbf{x}} + \frac{1}{r} G_2\left(\frac{\mathbf{b}^T \mathbf{x}}{r}\right) \right].$$

2. Integration of the system of equations in a special case

In this section we perform straightforward integration of a system with potential of the form (23) having additional parameters. Namely, consider the potential

$$V = V(\mathbf{x}) = -\frac{1}{r} \left(G_1((r + \mathbf{b}^T \mathbf{x})/2) + G_2((r - \mathbf{b}^T \mathbf{x})/2) \right), \quad (28)$$

where G_1, G_2 are some smooth functions, and $\mathbf{b} = (b_1, b_2, b_3)^T$ an arbitrary unit vector. Note that the vector \mathbf{b} provides two parameters in explicit form. Having in mind only theoretical investigation (integrability problem), one can take \mathbf{b} to be the ort along the x_1 -axis. On the other hand, from the more practical point of view, introducing vector \mathbf{b} gives us additional degree of freedom necessary for applied problems of celestial mechanics. In such problems, the axes are usually connected with some special directions (equinox or zenith). Therefore the presence of the vector \mathbf{b} in potential (28) allows one to turn the coordinate system at one's will.

As G_1, G_2 , one can take, for example, functions of the form

$$\frac{1}{r} (r + \mathbf{b}^T \mathbf{x})^k, \quad \frac{1}{r} (r - \mathbf{b}^T \mathbf{x})^k, \quad k = 1, 2, \dots$$

We consider a finite linear combination

$$V = -\frac{1}{r} \sum_{k=1}^N \left(A_k (r + \mathbf{b}^T \mathbf{x})^k + B_k (r - \mathbf{b}^T \mathbf{x})^k \right). \quad (29)$$

Here A_k, B_k are constants. Such a potential was considered in [16]. This case leads in general to hyperelliptic integrals.

For an interested reader there is a problem: find a real perturbing potential which can be approximated by functions of the form (29). Note that the combination

$$-\frac{B}{4r} (r + \mathbf{b}^T \mathbf{x})^2 + \frac{B}{4r} (r - \mathbf{b}^T \mathbf{x})^2 = -B \mathbf{b}^T \mathbf{x}$$

gives potential corresponding to a constant force. Applications of such potential were considered in [3–5].

The canonical equations of motion have the form

$$\begin{aligned} \frac{dx_i}{dt} &= y_i, \\ \frac{dy_i}{dt} &= -\frac{\mu}{r^3}x_i - \frac{x_i}{r^3}\left(G_1((r + \mathbf{b}^T \mathbf{x})/2) + \right. \\ &\quad \left. + G_2((r - \mathbf{b}^T \mathbf{x})/2)\right) + \\ &\quad + \frac{1}{2r}\left(G_1'((r + \mathbf{b}^T \mathbf{x})/2)\left(\frac{x_i}{r} + b_i\right) + \right. \\ &\quad \left. + G_2'(r - \mathbf{b}^T \mathbf{x})/2)\left(\frac{x_i}{r} - b_i\right)\right), \end{aligned} \quad (30)$$

where $i = 1, 2, 3$ and the sign prime indicates the derivative.

This system is the same as the equation of the perturbed two-body problem

$$\begin{aligned} \ddot{\mathbf{x}} + \frac{\mu}{r^3}\mathbf{x} &= \frac{1}{2r}\left(G_1'((r + \mathbf{b}^T \mathbf{x})/2) - \right. \\ &\quad \left. - G_2'((r - \mathbf{b}^T \mathbf{x})/2)\right)\mathbf{b} + \\ &\quad + \frac{1}{2r^2}\left(G_1'((r + \mathbf{b}^T \mathbf{x})/2) + \right. \\ &\quad \left. + G_2'((r - \mathbf{b}^T \mathbf{x})/2)\right)\mathbf{x} - \\ &\quad - \frac{1}{r^3}\left(G_1((r + \mathbf{b}^T \mathbf{x})/2) + \right. \\ &\quad \left. + G_2((r - \mathbf{b}^T \mathbf{x})/2)\right)\mathbf{x}. \end{aligned}$$

From this one can see that the perturbation is defined by two forces. The first force is collinear to the fixed vector \mathbf{b} , and its module varies in dependence on vector \mathbf{x} . The second force is the central one.

We are going to show that the system (30) is integrable in regular variables found by L -transformations. Transformation (8) contains an arbitrary L -matrix. A special choice of this matrix allows one to separate the variables in the case of an arbitrary constant unitary vector \mathbf{b} .

Let us consider the term in (10) containing $V_c(\mathbf{q})$. In the new variables this becomes

$$\begin{aligned} |\mathbf{q}|^2 V_c(\mathbf{q}) &= -G_1((|\mathbf{q}|^2 + \\ &\quad + \mathbf{q}^T(b_1 K_1 + b_2 K_2 + b_3 K_3)K_4 \mathbf{q})/2) - \\ &\quad - G_2((|\mathbf{q}|^2 - \mathbf{q}^T(b_1 K_1 + b_2 K_2 + b_3 K_3)K_4 \mathbf{q})/2). \end{aligned}$$

We assume that the L -matrix has the first type and $K_4 = -\mathcal{Y}$. Then

$$\begin{aligned} |\mathbf{q}|^2 V_c(\mathbf{q}) &= -G_1((|\mathbf{q}|^2 - C)/2) - \\ &\quad - G_2((|\mathbf{q}|^2 + C)/2), \end{aligned} \quad (31)$$

where

$$\begin{aligned} C &= \mathbf{q}^T \left[(b_1 a_{11} + b_2 a_{12} + b_3 a_{13})\mathcal{U} + \right. \\ &\quad \left. + (b_1 a_{21} + b_2 a_{22} + b_3 a_{23})\mathcal{V} + \right. \\ &\quad \left. + (b_1 a_{31} + b_2 a_{32} + b_3 a_{33})\mathcal{W} \right] \mathcal{Y} \mathbf{q}. \end{aligned}$$

Let's select parameters of L -matrixes a_{ij} from a system (17). Then

$$C = \mathbf{q}^T \mathcal{U} \mathcal{Y} \mathbf{q} = \mathbf{q}^T \begin{pmatrix} -q_1 \\ -q_2 \\ q_3 \\ q_4 \end{pmatrix} = -q_1^2 - q_2^2 + q_3^2 + q_4^2.$$

Substituting the found value C in (31), we obtain

$$|\mathbf{q}|^2 V_c(\mathbf{q}) = -G_1(q_1^2 + q_2^2) - G_2(q_3^2 + q_4^2).$$

It follows that the Hamiltonian (10) is represented in the form of the sum

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2,$$

where

$$\mathcal{K}_1 = \frac{1}{8}(p_1^2 + p_2^2) + p_0(q_1^2 + q_2^2) - G_1(q_1^2 + q_2^2),$$

$$\mathcal{K}_2 = \frac{1}{8}(p_3^2 + p_4^2) + p_0(q_3^2 + q_4^2) - G_2(q_3^2 + q_4^2).$$

As the value of p_0 is constant, the system (9) splits into two independent subsystems

$$\frac{dq_i}{d\tau} = \frac{\partial \mathcal{K}_1}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial \mathcal{K}_1}{\partial q_i}, \quad i = 1, 2, \quad (32)$$

$$\frac{dq_i}{d\tau} = \frac{\partial \mathcal{K}_2}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial \mathcal{K}_2}{\partial q_i}, \quad i = 3, 4. \quad (33)$$

We integrate the system (32) again. In the bipolar coordinates Hamiltonian \mathcal{K}_1 , and accordingly the system, have the form

$$\begin{aligned} \bar{\mathcal{K}}_1 &= \frac{1}{8}\left(4Q_1 P_1^2 + \frac{P_2^2}{Q_1}\right) + p_0 Q_1 - G_1(Q_1), \\ \frac{dQ_1}{d\tau} &= Q_1 P_1, \quad \frac{dQ_2}{d\tau} = \frac{P_2}{4Q_1}, \\ \frac{dP_1}{d\tau} &= -\frac{1}{2}P_1^2 + \frac{P_2^2}{8Q_1^2} - p_0 + G_1'(Q_1), \\ \frac{dP_2}{d\tau} &= 0. \end{aligned} \quad (34)$$

Since the Hamiltonian $\bar{\mathcal{K}}_1$ does not explicitly depend on τ and Q_2 , the system (34) has two integrals,

$$\begin{aligned} \frac{1}{2}Q_1 P_1^2 + \frac{P_2^2}{8Q_1} + p_0 Q_1 - G_1(Q_1) &= \frac{E_1}{8}, \\ P_2 &= c_1. \end{aligned} \quad (35)$$

Here, E_1 and c_1 are the constants of integration. Taking these integrals into account, the equation for P_1 may be written in the following form

$$\frac{dP_1}{d\tau} = \frac{c_1^2}{4Q_1^2} - \frac{E_1}{8Q_1} + G_1'(Q_1) - \frac{G_1(Q_1)}{Q_1}.$$

Eliminating $d\tau$ from equations for P_1 , Q_1 and integrating the resulting equation, we find

$$P_1 = \frac{\delta_1}{2Q_1} \sqrt{\Phi_1(Q_1)}, \quad \delta_1 = \pm 1,$$

where

$$\Phi_1(Q_1) = -c_1^2 + E_1 Q_1 + c_2 Q_1^2 + 8Q_1 G_1(Q_1)$$

and c_2 is integration constant defined by

$$c_2 = 4(P_1^0)^2 + \frac{c_1^2}{(Q_1^0)^2} - \frac{E_1}{Q_1^0} - 8 \frac{G_1(Q_1^0)}{Q_1^0}.$$

Due to nonnegativity of Q_1 , from the first equation of the system (34) it follows that

$$\delta_1 = \text{sign } Q_1'.$$

Substituting the derived P_1 to the first equation of (34) we find

$$\tau + c_3 = 2\delta_1 \int_{\xi}^{Q_1} \frac{dQ_1}{\sqrt{\Phi_1(Q_1)}}. \quad (36)$$

Using the continuity principle, the sign before the integral (36) cannot change when $\Phi_1(Q_1)$ is non-zero. Therefore, the function $\tau(Q_1)$ in this case behaves monotonically. Inverting the integral (36), we obtain Q_1 as a function of τ ; we substitute this function in the second equation of the system (34). Then we get

$$Q_2 = \frac{c_1}{4} \int_0^{\tau} \frac{d\tau}{Q_1(\tau)} + c_4, \quad c_4 = Q_2^0.$$

Here c_3 and c_4 are the integration constants. Thus, the values Q_1, Q_2, P_1 are represented as functions of τ . If $\Phi_1(Q_1)$ is a polynomial, the integral (36) is, in general, hyperelliptic.

The integration of the system (33) is done similarly. As a result we find

$$Q_4 = \frac{c_5}{4} \int_0^{\tau} \frac{d\tau}{Q_3(\tau)} + c_8, \quad c_8 = Q_4^0,$$

$$P_3 = \frac{\delta_2}{2Q_3} \sqrt{\Phi_2(Q_3)}, \quad \delta_2 = \text{sign } Q_3', \quad P_4 = c_5,$$

where

$$\Phi_2(Q_3) = -c_5^2 + E_2 Q_3 + c_6 Q_3^2 + 8Q_3 G_2(Q_3).$$

Here c_6 and E_2 are the integration constants defined by the equalities

$$c_6 = 4(P_3^0)^2 + \frac{c_5^2}{(Q_3^0)^2} - \frac{E_2}{Q_3^0} - 8 \frac{G_2(Q_3^0)}{Q_3^0},$$

$$\frac{E_2}{8} = \frac{1}{2} Q_3 P_3^2 + \frac{P_4^2}{8Q_3} + p_0 Q_3 - G_2(Q_3). \quad (37)$$

The function $Q_3(\tau)$ is found by a reversion of the integral

$$\tau + c_7 = 2\delta_2 \int_{\eta}^{Q_3} \frac{dQ_3}{\sqrt{\Phi_2(Q_3)}}. \quad (38)$$

Thus, the values $Q_3, Q_4,$ and P_3 are also determined as functions of the variable τ . The lower limits ξ and η in integrals (36) and (38) are chosen according to the location of Q_1 and Q_3 with respect to the roots of functions $\Phi_1(Q_1)$ and $\Phi_2(Q_3)$, respectively.

The formulae of inverse transformation

$$Q_1 = q_1^2 + q_2^2, \quad \tan Q_2 = \frac{q_2}{q_1},$$

$$Q_3 = q_3^2 + q_4^2, \quad \tan Q_4 = \frac{q_4}{q_3},$$

$$P_1 = \frac{q_1 p_1 + q_2 p_2}{2(q_1^2 + q_2^2)}, \quad P_2 = -q_2 p_1 + q_1 p_2,$$

$$P_3 = \frac{q_3 p_3 + q_4 p_4}{2(q_3^2 + q_4^2)}, \quad P_4 = -q_4 p_3 + q_3 p_4$$

allow to define initial values of the variables Q_i and P_i ($i = 1, 2, 3, 4$).

The values of integration constants $c_1, c_2, c_3, c_4,$ and E_1 are determined by the initial values of Q_1, Q_2, P_1, P_2 . These five constant values are connected with each other by the integral (35). In the same way, the constant values $c_5, c_6, c_7, c_8,$ and E_2 are connected by the integral (37) and are defined by initial values of $Q_3, Q_4, P_3,$ and P_4 . From $p_0 = -H(\mathbf{x}^0, \mathbf{y}^0)$ we also find relation $E_1 + E_2 = 8\mu$. One has to add the above relations for c_2 and c_6 to these connections. Besides, as the bilinear relation $\mathbf{q}^T K_4 \mathbf{p} = 0$ is the integral of (9), in our case we have

$$\mathbf{q}^T (-Y) \mathbf{p} = -q_2 p_1 + q_1 p_2 + q_4 p_3 - q_3 p_4 = 0.$$

Therefore the equality $P_2 = P_4$, or equivalently $c_1 = c_5$, also holds.

Applying further the first four formulas (19) and (20), we find q_i, p_i ($i = 1, 2, 3, 4$) as functions of τ . Finally, integrating the two remaining equations of (9), we obtain $p_0 = -H(\mathbf{x}^0, \mathbf{y}^0)$ and physical time expressed through τ ,

$$t = q_0 = \int_0^{\tau} |\mathbf{q}|^2 d\tau + c_9 = t_1 + t_2, \quad (39)$$

where

$$t_1 = \int_0^{\tau} Q_1(\tau) d\tau, \quad t_2 = \int_0^{\tau} Q_3(\tau) d\tau, \quad c_9 = 0.$$

Thus, the system (9) is completely integrated and we can, at least locally, find a required trajectory. Here it is necessary to note, that if perturbing potentials G_1, G_2 in (30) are analytic, then, as it is known from a course of the differential equations, the solution of the problem will also be analytic. Let us suppose that the local inversion of integrals (36), (38) appeared to be a globally determined function. In this case we can conclude, by uniqueness of analytic continuation, that this inversion gives not only local, but also global solution of the problem (30). This is the case when functions G_1, G_2 are polynomials of degree two or three. In this case (36) and (38) are the elliptic integrals, for which inversion we have the well developed technique of elliptic functions at our disposal; thus, we have found the solution of (30) in explicit form.

3. Inversion of the integral in elliptic case

In this section we consider one case of functions G_1 and G_2 , which reduces to elliptic integrals. Other cases have been studied in [7, 14, 16, 17]. Let us take the functions G_1 and G_2

$$G_1 = \frac{A_{-1}}{r + \mathbf{b}^T \mathbf{x}} + A_1(r + \mathbf{b}^T \mathbf{x}) + A_2(r + \mathbf{b}^T \mathbf{x})^2, \quad (40)$$

$$G_2 = \frac{B_{-1}}{r - \mathbf{b}^T \mathbf{x}} + B_1(r - \mathbf{b}^T \mathbf{x}) + B_2(r - \mathbf{b}^T \mathbf{x})^2, \quad (41)$$

where $A_{-1}, A_1, A_2, B_{-1}, B_1,$ and B_2 are the parameters of the potential. Then for the functions $\Phi_1(Q_1)$ and $\Phi_2(Q_3)$ in (36) and (38) we have the expressions

$$\Phi_1(Q_1) = \hat{c}_1 + E_1 Q_1 + c_2 Q_1^2 + 32A_2 Q_1^3,$$

$$\Phi_2(Q_3) = \hat{c}_5 + E_2 Q_3 + c_6 Q_3^2 + 32B_2 Q_3^3.$$

Here $\hat{c}_1 = -c_1^2 + 4A_{-1}, c_2 = 16A_1 - 8p_0, \hat{c}_5 = -c_5^2 + 4B_{-1}, c_6 = 16B_1 - 8p_0.$

Firstly, let us note that the variables Q_1 and Q_3 are non-negative by definition, and that from integrals (36) and (38) it follows that the ranges of these variables are determined by the inequalities

$$\Phi_1(Q_1) \geq 0, \quad \Phi_2(Q_3) \geq 0. \quad (42)$$

Let us reverse the integral (36). The number of roots of the polynomial Φ_1 and their positions depend on the value of A_2 . With $A_2 = 0$ the degree of $\Phi_1(Q_1)$ equals to two. The integral (36) is found in elementary functions, so this case is not considered here. We distinguish two cases: $A_2 < 0, A_2 > 0$. Let's note the roots of $\Phi_1(Q_1)$ as ξ_1, ξ_2, ξ_3 . The cases under consideration will be sequentially numbered by parameter i_A .

I. Assume that $A_2 < 0$. In this case $\Phi_1(-\infty) > 0, \Phi_1(+\infty) < 0$. The value $\Phi_1(0) = \hat{c}_1$ may be both positive and negative. For actual motion there should be at least one positive root. The qualitatively different cases of the graph of $\Phi_1(Q_1)$ are shown in Fig. 1 and 2. In the case of three real roots (Fig. 2), the axis of ordinates goes between ξ_1, ξ_2 if $\hat{c}_1 < 0$, and left with respect to ξ_1 or between ξ_2, ξ_3 if $\hat{c}_1 > 0$.

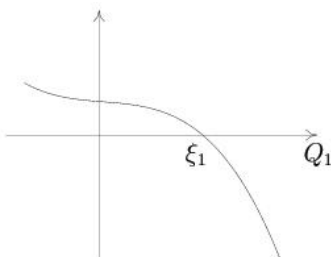


Fig. 1. The graph of $\Phi_1(Q_1)$. The case $A_2 < 0$.
Рис. 1. График $\Phi_1(Q_1)$. Случай $A_2 < 0$.

The case $i_A = 1$. Suppose that Φ_1 has one real root ξ_1 , and that $Q_1^0 \in (0, \xi_1)$ (Fig. 1). Let's write the integral (36) in the form

$$\tau + c_3 = \frac{\delta_1}{2\sqrt{-2A_2}} \int_{\xi_1}^{Q_1} \frac{dz}{\sqrt{(\xi_1 - z)(z^2 + bz + c)}},$$

where the square trinomial $z^2 + bz + c$ has no real roots and is positive for all z , and

$$b = \xi_1 + \frac{c_2}{32A_2}, \quad c = b\xi_1 + \frac{E_1}{32A_2} \quad (c > 0). \quad (43)$$

Apply the substitution

$$z = \xi_1 - a \frac{1 - \cos \varphi}{1 + \cos \varphi}, \quad a = \sqrt{\xi_1^2 + b\xi_1 + c}$$

in the integral and put the notations

$$\varphi_1 = 2 \arctan \sqrt{\frac{\xi_1 - Q_1}{a}}, \quad k_1^2 = \frac{1}{2} \left(1 + \frac{\xi_1 + b/2}{a} \right),$$

$$l_1 = 2\sqrt{-2aA_2}.$$

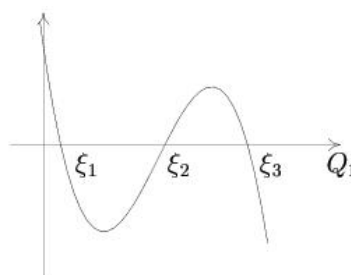


Fig. 2. The graph of $\Phi_1(Q_1)$. The case $A_2 < 0$.
Рис. 2. График $\Phi_1(Q_1)$. Случай $A_2 < 0$.

Then we derive

$$\tau + c_3 = -\frac{\delta_1}{l_1} \int_0^{\varphi_1} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}}. \quad (44)$$

Putting here $\tau = 0$, we find an integration constant c_3 :

$$c_3 = -\frac{\text{sign } P_1^0}{l_1} \int_0^{\varphi_1^0} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}},$$

$$\varphi_1^0 = 2 \arctan \sqrt{\frac{\xi_1 - Q_1^0}{a}}.$$

Check that $k_1^2 < 1$. As $z^2 + bz + c$ has no real roots, we have $b^2 - 4c < 0$. Therefore,

$$\left(\xi_1 + \frac{b}{2} \right)^2 < \xi_1^2 + b\xi_1 + c = a^2 \Rightarrow \left| \frac{\xi_1 + b/2}{a} \right| < 1.$$

Hence, $|k_1| < 1$. Reversing the integral (44) derived above, we come to the function

$$Q_1 = \xi_1 + a - \frac{2a}{1 + \text{cn}(l_1(\tau + c_3); k_1)}.$$

It is easy to see that for $Q_1 \in (0, \xi_1)$ the denominator $\text{cn}(u) + 1 \neq 0$. Calculating the derivative of Q_1 , we get $\delta_1 = -\text{sign } \text{sn}(l_1(\tau + c_3); k_1)$. For the variable Q_2 we

find

$$Q_2 = \frac{c_1\tau}{4(\xi_1 + a)} + \frac{ac_1}{2l_1(\xi_1^2 - a^2)} \left[\int_0^{l_1(\tau+c_3)} \frac{du}{1 + n_1 \operatorname{cn}(u; k_1)} - \int_0^{l_1c_3} \frac{du}{1 + n_1 \operatorname{cn}(u; k_1)} \right] + Q_2^0,$$

$$n_1 = 1 + \frac{2a}{\xi_1 - a}.$$

Note that

$$\frac{n_1^2}{n_1^2 - 1} - k_1^2 = \frac{c}{4a\xi_1} > 0.$$

Therefore for calculating the integral of $(1 + n_1 \operatorname{cn}(u; k_1))^{-1}$ we apply the formula (341.03) [15]

$$\int_0^u \frac{du}{1 + n \operatorname{cn}(u; k)} = \frac{1}{1 - n^2} \left[\Pi\left(u, \frac{n^2}{n^2 - 1}; k\right) - ng_1 \right],$$

$$n^2 \neq 1, \quad (45)$$

with

$$g_1(u) = \frac{1}{2} \sqrt{\frac{n^2 - 1}{k^2 + k'^2 n^2}} \times \ln \left| \frac{\sqrt{k^2 + k'^2 n^2} \operatorname{sn}(u; k) + \sqrt{n^2 - 1} \operatorname{dn}(u; k)}{\sqrt{k^2 + k'^2 n^2} \operatorname{sn}(u; k) - \sqrt{n^2 - 1} \operatorname{dn}(u; k)} \right|,$$

$$k'^2 = 1 - k^2.$$

Here we note the elliptic integral of the third kind as

$$\Pi(u, n; k) \equiv \int_0^u \frac{dv}{1 - n \operatorname{sn}^2(v; k)}.$$

For t_1 we have

$$t_1 = (\xi_1 + a)\tau - \frac{2a}{l_1} \left[\int_0^{l_1(\tau+c_3)} \frac{du}{1 + \operatorname{cn}(u; k_1)} - \int_0^{l_1c_3} \frac{du}{1 + \operatorname{cn}(u; k_1)} \right].$$

The integral of $(1 + \operatorname{cn}(u; k_1))^{-1}$ is calculated by the formula (341.53) [15]

$$\int_0^u \frac{dv}{1 \pm \operatorname{cn}(v; k)} = u - E(u) \pm \frac{\operatorname{dn}(u; k) \operatorname{sn}(u; k)}{1 \pm \operatorname{cn}(u; k)}, \quad (46)$$

where $E(u) = E(\varphi; k)$ is incomplete elliptic integral of the second kind ($\varphi = \operatorname{am} u$).

The case $i_A = 2$. Suppose that $\Phi_1(Q_1)$ has three real roots $0 < \xi_1 < \xi_2 < \xi_3$, and $Q_1^0 \in (0, \xi_1)$. Let's write (36) as

$$\tau + c_3 = \frac{\delta_1}{2\sqrt{-2A_2}} \int_{Q_1}^{\xi_1} \frac{dz}{\sqrt{(\xi_1 - z)(\xi_2 - z)(\xi_3 - z)}}.$$

Making the substitution $\varphi = \arcsin \sqrt{(\xi_1 - z)/(\xi_2 - z)}$ and reversing the resulting integral, we find

$$Q_1 = \xi_2 - \frac{(\xi_2 - \xi_1)}{\operatorname{cn}^2(l_1(\tau + c_3); k_1)},$$

where

$$k_1 = \sqrt{\frac{\xi_3 - \xi_2}{\xi_3 - \xi_1}}, \quad l_1 = \sqrt{-2A_2(\xi_3 - \xi_1)},$$

$$c_3 = \frac{\operatorname{sign} P_1^0}{l_1} \int_0^{\varphi_1^0} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}},$$

$$\varphi_1^0 = \arcsin \sqrt{\frac{\xi_1 - Q_1^0}{\xi_2 - Q_1^0}}.$$

Now we calculate δ_1 . We differentiate Q_1 and use the formula of double argument for elliptic sine. We have

$$Q_1' = 2l_1(\xi_3 - \xi_2) \operatorname{cn}^{-3}(u; k_1) (-1) \operatorname{sn}(u; k_1) \operatorname{dn}(u; k_1) = -l_1(\xi_3 - \xi_2) \operatorname{cn}^{-4}(u; k_1) (1 - k_1^2 \operatorname{sn}^4(u; k_1)) \operatorname{sn}(2u; k_1),$$

where the notation $u = l_1(\tau + c_3)$ is introduced for brevity. Therefore,

$$\delta_1 = \operatorname{sign} Q_1' = -\operatorname{sign} \operatorname{sn}(2l_1(\tau + c_3); k_1).$$

Now we find Q_2

$$Q_2 = \frac{c_1\tau}{4\xi_2} + \frac{c_1(\xi_2 - \xi_1)}{4l_1\xi_1\xi_2} \left[\Pi(l_1(\tau + c_3), n_1; k_1) - \Pi(l_1c_3, n_1; k_1) \right] + Q_2^0, \quad n_1 = \frac{\xi_2}{\xi_1}.$$

For the value of physical time, corresponding to the variable Q_1 , we have

$$t_1 = \xi_2\tau + \frac{\xi_1 - \xi_2}{l_1} \left[\int_0^{l_1(\tau+c_3)} \frac{du}{\operatorname{cn}^2(u; k_1)} - \int_0^{l_1c_3} \frac{du}{\operatorname{cn}^2(u; k_1)} \right],$$

where the integral of $\operatorname{cn}^{-2}(u; k_1)$ is calculated by the formula (313.02) [15]

$$\int_0^u \frac{dv}{\operatorname{cn}^2(v; k)} = \frac{1}{1 - k^2} \left((1 - k^2)u - E(u) + \frac{\operatorname{dn}(u; k) \operatorname{sn}(u; k)}{\operatorname{cn}(u; k)} \right).$$

The case $i_A = 3$. The polynomial $\Phi_1(Q_1)$ has three real roots $\xi_1 < \xi_2 < \xi_3$, and $Q_1^0 \in (\max\{0, \xi_2\}, \xi_3)$. Let us write (36) as

$$\tau + c_3 = \frac{\delta_1}{2\sqrt{-2A_2}} \int_{\xi_3}^{Q_1} \frac{dz}{\sqrt{(z - \xi_1)(z - \xi_2)(\xi_3 - z)}}.$$

The reduction of this integral to the standard form (44) is carried out by the substitution $\varphi = \arcsin \sqrt{(\xi_3 - z)/(\xi_3 - \xi_2)}$. The result of reversion can be presented in the form

$$Q_1 = \xi_3 + (\xi_2 - \xi_3) \operatorname{sn}^2(l_1(\tau + c_3); k_1),$$

where the following notations are used

$$k_1 = \sqrt{\frac{\xi_3 - \xi_2}{\xi_3 - \xi_1}}, \quad l_1 = \sqrt{-2A_2(\xi_3 - \xi_1)},$$

$$c_3 = -\frac{\operatorname{sign} P_1^0}{l_1} \int_0^{\varphi_1^0} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}},$$

$$\varphi_1^0 = \arcsin \sqrt{\frac{\xi_3 - Q_1^0}{\xi_3 - \xi_2}}.$$

For δ_1 we find $\delta_1 = -\operatorname{sign} \operatorname{sn}(2l_1(\tau + c_3); k_1)$. Substitute Q_1 in the formulae for Q_2 and t_1 . We find

$$Q_2 = \frac{c_1}{4l_1\xi_3} \left[\Pi(l_1(\tau + c_3), n_1; k_1) - \Pi(l_1c_3, n_1; k_1) \right] + Q_2^0,$$

$$t_1 = \xi_3\tau + \frac{\xi_2 - \xi_3}{l_1} \left[\int_0^{l_1(\tau+c_3)} \operatorname{sn}^2(u; k_1) du - \int_0^{l_1c_3} \operatorname{sn}^2(u; k_1) du \right].$$

The integral of squared elliptic sine is calculated by the formula [15]

$$\int_0^u \operatorname{sn}^2(v; k) dv = \frac{1}{k^2}(u - E(u)).$$

II. Assume further that $A_2 > 0$. Now we have $\Phi_1(-\infty) < 0$, $\Phi_1(+\infty) > 0$, and $\Phi_1(0) = \hat{c}_1$. The qualitatively different cases of the graph $\Phi_1(Q_1)$ are shown in Fig. 3 and 4.

The case $i_A = 4$. The polynomial $\Phi_1(Q_1)$ has one real root ξ_1 and, accordingly, $Q_1(0) > \max\{0, \xi_1\}$. The graph of $\Phi_1(Q_1)$ in this case is shown in Fig. 3. Let us write the integral (36) as

$$\tau + c_3 = \frac{\delta_1}{2\sqrt{2A_2}} \int_{\xi_1}^{Q_1} \frac{dz}{\sqrt{(z - \xi_1)(z^2 + bz + c)}},$$

where the square trinomial $z^2 + bz + c > 0$ for all z . The coefficients b and c are found by the formulae (43). Applying the substitution

$$z = \xi_1 + a \frac{1 - \cos \varphi}{1 + \cos \varphi}, \quad a = \sqrt{\xi_1^2 + b\xi_1 + c}$$

and reversing the resulting integral, we come to the function

$$Q_1 = \xi_1 - a + \frac{2a}{1 + \operatorname{cn}(l_1(\tau + c_3); k_1)},$$

where

$$k_1^2 = \frac{1}{2} \left(1 - \frac{\xi_1 + b/2}{a} \right), \quad l_1 = 2\sqrt{2aA_2}.$$

$$c_3 = \frac{\operatorname{sign} P_1^0}{l_1} \int_0^{\varphi_1^0} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}},$$

$$\varphi_1^0 = 2 \arctan \sqrt{\frac{Q_1^0 - \xi_1}{a}}.$$

As above, one can show that $k_1^2 < 1$. The resulting function $Q_1(\tau)$ is unbounded, as it has an infinite number of poles on real straight line, which are found by the formula

$$\tau = \frac{4m + 2}{l_1} K(k_1) - c_3, \quad m \in \mathbf{Z}.$$

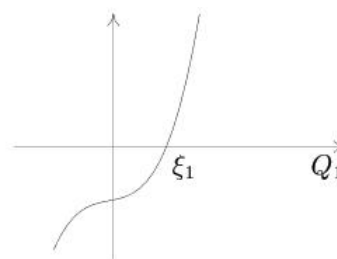


Fig. 3. The graph $\Phi_1(Q_1)$. Case $A_2 > 0$.
Рис. 3. График $\Phi_1(Q_1)$. Случай $A_2 > 0$.

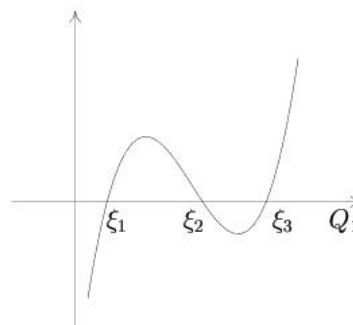


Fig. 4. The graph $\Phi_1(Q_1)$. Case $A_2 > 0$.
Рис. 4. График $\Phi_1(Q_1)$. Случай $A_2 > 0$.

Further we find that $\delta_1 = \text{sign sn}(l_1(\tau + c_3); k_1)$. For variable Q_2 we have

$$Q_2 = \frac{c_1\tau}{4(\xi_1 - a)} - \frac{ac_1}{2l_1(\xi_1^2 - a^2)} \left[\int_0^{l_1(\tau+c_3)} \frac{du}{1 + n_1 \text{cn}(u; k_1)} - \int_0^{l_1c_3} \frac{du}{1 + n_1 \text{cn}(u; k_1)} \right] + Q_2^0,$$

$$n_1 = 1 - \frac{2a}{\xi_1 + a}.$$

Note that

$$\frac{n_1^2}{n_1^2 - 1} = 1 + \frac{1}{n_1^2 - 1} = -\frac{(\xi_1 - a)^2}{4a\xi_1} < 0 < k_1^2.$$

Therefore for calculating the integral of the function $(1 + n_1 \text{cn}(u; k_1))^{-1}$ the formula (45) is to be applied with

$$g_1 = \sqrt{\frac{1 - n^2}{k^2 + k'^2 n^2}} \arctan \left[\sqrt{\frac{k^2 + k'^2 n^2}{1 - n^2}} \frac{\text{sn}(u; k)}{\text{dn}(u; k)} \right],$$

$$k'^2 = 1 - k^2.$$

If $\xi_1 = 0$ then $n_1 = -1$, and for calculating Q_2 the formula (46) should be used. For t_1 we have

$$t_1 = (\xi_1 - a)\tau + \frac{2a}{l_1} \left[\int_0^{l_1(\tau+c_3)} \frac{du}{1 + \text{cn}(u; k_1)} - \int_0^{l_1c_3} \frac{du}{1 + \text{cn}(u; k_1)} \right].$$

Suppose that Φ_1 has three real roots $\xi_1 < \xi_2 < \xi_3$. The graph of the function $\Phi_1(Q_1)$ in this case is given in Fig. 4. This case also splits into two subcases: $\xi_1 < Q_1^0 < \xi_2$ and $\xi_3 < Q_1^0$.

The case $i_A = 5$. Suppose that $Q_1^0 \in (\max\{0, \xi_1\}, \xi_2)$. We write (36) as

$$\tau + c_3 = \frac{\delta_1}{2\sqrt{2A_2}} \int_{\xi_1}^{Q_1} \frac{dz}{\sqrt{(z - \xi_1)(z - \xi_2)(z - \xi_3)}}.$$

We apply the substitution $\varphi = \arcsin \sqrt{(z - \xi_1)/(\xi_2 - \xi_1)}$ to this integral and use the notations

$$k_1 = \sqrt{\frac{\xi_2 - \xi_1}{\xi_3 - \xi_1}}, \quad l_1 = \sqrt{2A_2(\xi_3 - \xi_1)}.$$

Then our integral has the standard form

$$\tau + c_3 = \frac{\delta_1}{l_1} \int_0^{\varphi_1} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}}. \quad (47)$$

Reversing (47) and using the inverse substitution, we find the required function

$$Q_1 = \xi_1 + (\xi_2 - \xi_1) \text{sn}^2(l_1(\tau + c_3); k_1),$$

where

$$c_3 = \frac{\text{sign } P_1^0}{l_1} \int_0^{\varphi_1^0} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}},$$

$$\varphi_1^0 = \arcsin \sqrt{\frac{Q_1^0 - \xi_1}{\xi_2 - \xi_1}}.$$

One can show that $\delta_1 = \text{sign sn}(2l_1(\tau + c_3); k_1)$. For Q_2 we find

$$Q_2 = \frac{c_1}{4l_1\xi_1} \left[\Pi(l_1(\tau + c_3), n_1; k_1) - \Pi(l_1c_3, n_1; k_1) \right] + Q_2^0, \quad n_1 = 1 - \frac{\xi_2}{\xi_1}.$$

For the first summand of physical time t in (39) we have

$$t_1 = \xi_1\tau + \frac{\xi_2 - \xi_1}{l_1} \left[\int_0^{l_1(\tau+c_3)} \text{sn}^2(u; k_1) du - \int_0^{l_1c_3} \text{sn}^2(u; k_1) du \right].$$

Let us consider the case $i_A = 6$. Suppose $Q_1^0 \in (\max\{0, \xi_3\}, \infty)$. The integral (36) has the form

$$\tau + c_3 = \frac{\delta_1}{2\sqrt{2A_2}} \int_{\xi_3}^{Q_1} \frac{dz}{\sqrt{(z - \xi_1)(z - \xi_2)(z - \xi_3)}}.$$

Using the substitution $\varphi = \arcsin \sqrt{(z - \xi_3)/(z - \xi_2)}$ we transform this integral to the standard form (47). The resulting reversion of the integral in this case is the following

$$Q_1 = \xi_2 + \frac{\xi_3 - \xi_2}{\text{cn}^2(l_1(\tau + c_3); k_1)},$$

where

$$k_1 = \sqrt{\frac{\xi_2 - \xi_1}{\xi_3 - \xi_1}}, \quad l_1 = \sqrt{2A_2(\xi_3 - \xi_1)},$$

$$c_3 = \frac{\text{sign } P_1^0}{l_1} \int_0^{\varphi_1^0} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}},$$

$$\varphi_1^0 = \arcsin \sqrt{\frac{Q_1^0 - \xi_3}{Q_1^0 - \xi_2}}.$$

Now the function $Q_1(\tau)$ has an infinite number of poles of the second order, hence it is unbounded. The poles are found by the formula

$$\tau = \frac{2m + 1}{l_1} K(k_1) - c_3, \quad m \in \mathbf{Z}.$$

Further we find the values δ_1, Q_2, t_1

$$\delta_1 = \text{sign sn}(2l_1(\tau + c_3); k_1),$$

$$Q_2 = \frac{c_1\tau}{4\xi_2} + \frac{c_1(\xi_2 - \xi_3)}{4l_1\xi_2\xi_3} \left[\Pi(l_1(\tau + c_3), n_1; k_1) - \Pi(l_1c_3, n_1; k_1) \right] + Q_2^0, \quad n_1 = \frac{\xi_2}{\xi_3},$$

$$t_1 = \xi_2\tau + \frac{\xi_3 - \xi_2}{l_1} \left[\int_0^{l_1(\tau+c_3)} \frac{du}{\text{cn}^2(u; k_1)} - \int_0^{l_1c_3} \frac{du}{\text{cn}^2(u; k_1)} \right].$$

An inversion of the integral (38) is fulfilled in a similar way. This integral and the function Φ_2 differ only by notations from the integral (36) and the function Φ_1 . Therefore, after some evident renaming, we find the expressions for Q_3, Q_4 , and t_2 . We number these cases sequentially by the parameter i_B . Then we have:

$$i_B = 1 \Leftrightarrow B_2 < 0, \eta_2, \eta_3 \in \mathbf{C}, 0 < Q_3^0 < \eta_1 \quad (Q_3 \text{ is bounded}).$$

$$i_B = 2 \Leftrightarrow B_2 < 0, 0 < Q_3^0 < \eta_1 < \eta_2 < \eta_3 \quad (Q_3 \text{ is bounded}).$$

$$i_B = 3 \Leftrightarrow B_2 < 0, \eta_1 < \eta_2 < Q_3^0 < \eta_3 \quad (Q_3 \text{ is bounded}).$$

$$i_B = 4 \Leftrightarrow B_2 > 0, \eta_2, \eta_3 \in \mathbf{C}, \eta_1 < Q_3^0 \quad (Q_3 \text{ is unbounded}).$$

$$i_B = 5 \Leftrightarrow B_2 > 0, \eta_1 < Q_3^0 < \eta_2 < \eta_3 \quad (Q_3 \text{ is bounded}).$$

$$i_B = 6 \Leftrightarrow B_2 > 0, \eta_1 < \eta_2 < \eta_3 < Q_3^0 \quad (Q_3 \text{ is unbounded}).$$

The study above yield the following theorem.

Theorem 3. *The motion of the particle is bounded if and only if at the initial moment both variables Q_1 and Q_3 are restricted on the right by the roots of the polynomials Φ_1 and Φ_2 , correspondingly.*

Now we give a definition of retaining potential, introduced in [14].

Definition 1. A potential is named as *retaining*, if for arbitrary initial conditions the motion of a particle in a perturbed field corresponding to this potential is bounded.

Thus, potential (28), where G_1, G_2 are defined by the formulae (40), (41) for $A_2 < 0$ and $B_2 < 0$, is retaining. Generally, the formulae (36), (38) are not elliptic integrals, and we cannot present a solution in explicit form. Nevertheless, the above-stated qualitative result remains true [16].

4. Numerical examples and analysis of motions

In the examples below we consider the motion of a particle in perturbed gravitational field of a planet with

spherical density distribution, which gravitational parameter is taken to be $\mu = 398601.3 \text{ km}^3/\text{s}^2$. The perturbing force is defined by the potential (28), with G_1 and G_2 calculated by the formulae (40) and (41). For convenience (to have no fractions), a dimensionless direction vector $\hat{\mathbf{b}}$ for the constant force is used. While doing calculations, this direction vector is assumed to be normalized. The dimensions of parameters A_{-1} and B_{-1} are $[\text{km}^4/\text{s}^2]$, A_1 and B_1 are $[\text{km}^2/\text{s}^2]$, A_2 and B_2 are $[\text{km}/\text{s}^2]$. Calculations and construction of orbits were performed using the Maple system with 32 digits. In each example, for convenience of its analysis, the values of circular and parabolic velocities v_{cir}, v_{par} of Keplerian motion are given. The perturbations under consideration are great, they are non-typical for the Earth's satellites. For this reason, we do not give Keplerian elements of oscillating orbits for the corresponding initial values. The initial position of a particle is marked by a point on the corresponding figure.

Example 1. Initial values of coordinates and velocities of a particle:

$$x_1 = 8200 \text{ km}, \quad x_2 = 0 \text{ km}, \quad x_3 = 6000 \text{ km},$$

$$\dot{x}_2 = 8.6 \text{ km/s}, \quad \dot{x}_1 = \dot{x}_3 = 0 \text{ km/s},$$

$$(v_{cir} \approx 6.26 \text{ km/s}, \quad v_{par} \approx 8.86 \text{ km/s}).$$

In an unperturbed case these values define an elliptic motion.

Parameters of potential are as follows:

$$A_{-1} = 0.004 \text{ km}^4/\text{s}^2, \quad A_1 = 0.06 \text{ km}^2/\text{s}^2,$$

$$A_2 = 0.2 \cdot 10^{-7} \text{ km}/\text{s}^2, \quad B_{-1} = 0.0001 \text{ km}^4/\text{s}^2,$$

$$B_1 = 0.008 \text{ km}^2/\text{s}^2, \quad B_2 = -0.3 \cdot 10^{-4} \text{ km}/\text{s}^2.$$

Coordinates of direction vector are $\hat{\mathbf{b}} = (-1, 2, 1)^T$. In the case under consideration the roots of polynomials Φ_1 and Φ_2 are

$$\xi_1 \approx 1478, \quad \xi_2 \approx 115346,$$

$$Q_1^0 \approx 4631 \Rightarrow Q_1^0 \in (\xi_1, \xi_2),$$

$$\eta_2 \approx 1707, \quad \eta_3 \approx 31031,$$

$$Q_3^0 \approx 5529 \Rightarrow Q_3^0 \in (\eta_2, \eta_3).$$

Therefore, the motion is bounded. This is the case $i_A = 5, i_B = 3$.

The coordinates and velocities have been calculated during a time range, corresponding to two revolutions of the particle around the attracting centre without perturbations, that is $\tau \in [0, 2T]$, where T is calculated by the formula

$$T = \pi \sqrt{-\frac{2}{h_k}}, \quad h_k = \frac{|\dot{\mathbf{x}}^0|^2}{2} - \frac{\mu}{|\mathbf{x}^0|}. \quad (48)$$

Here h_k is the Keplerian energy. Let's remind that L -transformation doubles the angles at the origin of coordinates.

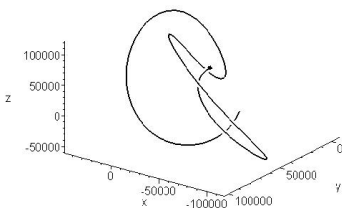


Fig. 5. The case $i_A = 5, i_B = 3$.
Рис. 5. Случай $i_A = 5, i_B = 3$.

Note that in this example the potential is not retaining. Nevertheless, the motion appears to be bounded. The trajectory of the particle is shown in Fig. 5.

Example 2. Initial values of coordinates and velocities of a particle:

$$\begin{aligned} x_1 &= 8200 \text{ km}, & x_2 &= 0 \text{ km}, & x_3 &= 6000 \text{ km}, \\ \dot{x}_2 &= 9.9 \text{ km/s}, & \dot{x}_1 &= \dot{x}_3 = 0 \text{ km/s}, \\ (v_{cir} &\approx 6.26 \text{ km/s}, & v_{par} &\approx 8.86 \text{ km/s}). \end{aligned}$$

In unperturbed case the motion belongs to hyperbolic type.

Parameters of potential are as follows:

$$\begin{aligned} A_{-1} &= 0.004 \text{ km}^4/\text{s}^2, & A_1 &= 0.006 \text{ km}^2/\text{s}^2, \\ A_2 &= -0.2 \cdot 10^{-7} \text{ km}/\text{s}^2, & B_{-1} &= 0.0001 \text{ km}^4/\text{s}^2, \\ B_1 &= 0.008 \text{ km}^2/\text{s}^2, & B_2 &= -0.3 \cdot 10^{-7} \text{ km}/\text{s}^2. \end{aligned}$$

As A_2 and B_2 are negative we have a retaining potential. Coordinates of direction vector are $\hat{\mathbf{b}} = (1, 2, -1)^T$. The roots of polynomials:

$$\begin{aligned} \xi_2 &\approx 2126, & \xi_3 &\approx 122192633, \\ Q_1^0 &\approx 5529 \Rightarrow Q_1^0 \in (\xi_2, \xi_3), \\ \eta_2 &\approx 1699, & \eta_3 &\approx 81506371, \\ Q_3^0 &\approx 4631 \Rightarrow Q_3^0 \in (\eta_2, \eta_3). \end{aligned}$$

Therefore, the motion is bounded. This is the case $i_A = 3, i_B = 3$. The integration is carried out during the time range corresponding approximately to $t = 1759.74$ days. The particle trajectory is shown in Fig. 6.

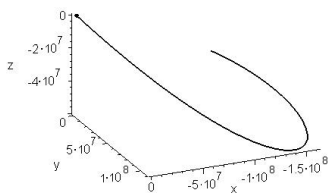


Fig. 6. The case $i_A = 3, i_B = 3$.
Рис. 6. Случай $i_A = 3, i_B = 3$.

Example 3. Initial values of coordinates and velocities of a particle are as follows:

$$\begin{aligned} x_1 &= 6000 \text{ km}, & x_2 &= 0 \text{ km}, & x_3 &= -8000 \text{ km}, \\ \dot{x}_2 &= 7.9 \text{ km/s}, & \dot{x}_1 &= \dot{x}_3 = 0 \text{ km/s}, \\ (v_{cir} &\approx 6.31 \text{ km/s}, & v_{par} &\approx 8.93 \text{ km/s}). \end{aligned}$$

In an unperturbed case these values define an elliptic motion.

Parameters of a potential are as follows:

$$\begin{aligned} A_{-1} &= 0.04 \text{ km}^4/\text{s}^2, & A_1 &= 0.03 \text{ km}^2/\text{s}^2, \\ A_2 &= -0.2 \cdot 10^{-5} \text{ km}/\text{s}^2, & B_{-1} &= 0.1 \cdot 10^{-4} \text{ km}^4/\text{s}^2, \\ B_1 &= -0.0003 \text{ km}^2/\text{s}^2, & B_2 &= 0.3 \cdot 10^{-4} \text{ km}/\text{s}^2. \end{aligned}$$

Here the potential is not retaining. Coordinates of direction vector are $\hat{\mathbf{b}} = (1, 1, 1)^T$. The roots of polynomials are as follows:

$$\begin{aligned} \xi_2 &\approx 2686, & \xi_3 &\approx 20699, \\ Q_1^0 &\approx 4423 \Rightarrow Q_1^0 \in (\xi_2, \xi_3), \\ \eta_1 &\approx 3256, & \eta_2, \eta_3 &\in \mathbf{C}, \\ Q_3^0 &\approx 5577 \Rightarrow Q_3^0 > \eta_1. \end{aligned}$$

Therefore, the motion is unbounded. This is the case $i_A = 3, i_B = 4$.

The integration is carried during the time range corresponding approximately to $t = 3.23$ days. The particle trajectory is shown in Fig. 7.

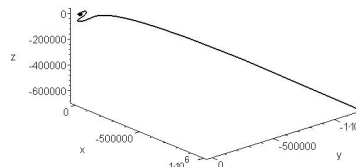


Fig. 7. The case $i_A = 3, i_B = 4$.
Рис. 7. Случай $i_A = 3, i_B = 4$.

In the case of unbounded motion, to define the integration interval firstly one has to find the nearest pole of $Q_1(\tau)$ and/or $Q_3(\tau)$ in the direction of ascending τ . Suppose this nearest pole is at $\tau = \tau_1$. Then we choose a small positive value ε and divide the segment $[0, \tau_1 - \varepsilon]$ into N equal subsegments. The value N is to be selected from practical reasons. The orbit should be visually a smooth curve. In our examples the value $N = 100$ was used. After that, the calculations by the formulae derived above are carried out in equidistant nodes.

The following example demonstrates an application of our formulae for testing a numerical integration method. The original system of motion equations (30) is considered. The Runge-Kutta-Fehlberg method of the eighth order with automatic choice of integration step is tested. The step is chosen by a method of the seventh order. The corresponding pair of programs, implemented in FORTRAN, is below noted as *RKF8(7)*. Integration of equations (30) was performed by *RKF8(7)* with relative local error of the method $\varepsilon = 10^{-13}$, and all calculations were carried out with double precision (real*8). The gravity parameter and the units of measurement are the same as above. A hypothetical particle is considered, repeatedly encountering the attracting centre. The trajectory obtained by explicit formulae is taken to be standard (reference). Its coordinates have been obtained using Maple with 32 digits (in FORTRAN this corresponds to quadruple precision (real*16)).

Example 4. Initial values of coordinates and velocities of a particle are as follows:

$$x_1 = 7000 \text{ km}, \quad x_2 = 0 \text{ km}, \quad x_3 = 6000 \text{ km},$$

Table 1

Estimation of the precision of numerical integration

Таблица 1

Оценка точности численного интегрирования

n	$t(\text{day})$	$\delta H \times 10^{-12}$	$\delta x_1 \times 10^{-12}$	$\delta x_2 \times 10^{-12}$	$\delta x_3 \times 10^{-12}$	$\delta r \times 10^{-12}$
1	.3382444	1	0.2	1	1	0.4
10	4.9080991	2	6	12	213	10
50	24.1940313	41	729	104	1667	108
100	48.4322508	53	4399798	523748	95154	330606
500	242.7821163	294	77898	31418	151259	77206
1000	485.2955201	556	554500	332688	1067003	330900

$$\dot{x}_2 = 7.9 \text{ km/s}, \quad \dot{x}_1 = \dot{x}_3 = 0 \text{ km/s},$$

$$(v_{cir} \approx 6.58 \text{ km/s}, \quad v_{par} \approx 9.30 \text{ km/s}).$$

In an unperturbed case we have an elliptic motion. Parameters of retaining potential are as follows:

$$A_{-1} = 0.1 \text{ km}^4/\text{s}^2, \quad A_1 = -0.02 \text{ km}^2/\text{s}^2,$$

$$A_2 = -0.2 \cdot 10^{-5} \text{ km}/\text{s}^2, \quad B_{-1} = -0.004 \text{ km}^4/\text{s}^2,$$

$$B_1 = -0.001 \text{ km}^2/\text{s}^2, \quad B_2 = -0.001 \text{ km}/\text{s}^2.$$

Coordinates of direction vector are $\hat{\mathbf{b}} = (-1, -3, 1)^T$. The roots of polynomials Φ_1 and Φ_2 are as follows:

$$\xi_2 \approx 764, \quad \xi_3 \approx 58639,$$

$$Q_1^0 \approx 4459 \Rightarrow Q_1^0 \in (\xi_2, \xi_3),$$

$$\eta_2 \approx 504, \quad \eta_3 \approx 7209,$$

$$Q_3^0 \approx 4761 \Rightarrow Q_3^0 \in (\eta_2, \eta_3).$$

Therefore, the motion is bounded. The case $i_A = 3, i_B = 3$.

The calculations were carried out during the time ranges corresponding to 1, 10, 50, 100, 500, and 1000 revolutions of the particle around attracting centre without perturbations. The trajectory of the particle for three revolutions is shown in Fig. 8.

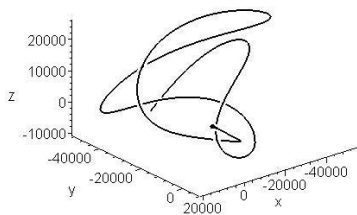


Fig. 8. The case $i_A = 3, i_B = 3$. The motion is bounded. Рис. 8. Случай $i_A = 3, i_B = 3$. Движение ограничено.

Table 1 contains the values, near the end of the integration interval, of the relative error for the energy constant δH , the coordinates of particle position vector x_i , and its absolute value r

$$\delta H = \frac{|H^0 - H|}{|H|}, \quad \delta x_i = \frac{|x_i - x_i^c|}{|x_i|}, \quad i = 1, 2, 3,$$

$$\delta r = \frac{|r - r^c|}{r},$$

where H^0 is the value at the initial moment, H at an arbitrary moment; x_i^c, r^c are the values found by exact formulae. In the second column the intervals of physical time t (in days) are given, for which numerical integration of system (30) was carried out.

From these data we can see that if the integration interval increases, the relative errors of H and x_3 do not decrease. For coordinates x_1, x_2 , and absolute value r , with $n = 100$, these errors increase, then they diminish, and then increase again.

The numerical examples show efficiency of the formulae we obtained. Besides, the theorem 3 allows to determine, given the initial position and velocity of a particle, whether its motion is bounded or unbounded.

Conclusion

In this paper we consider three sorts of coordinates (regular q -coordinates, bipolar coordinates, spherical coordinates). For each of the systems, the forms of potentials admitting complete separation of variables are given. Thus, the original equations for such potentials allow integration “in the sense of Sundman”. In a similar way one can build, for regular q -coordinates, other coordinate systems for which Hamiltonian has orthogonal form, and with the use of Stackel theorem build potentials allowing the above-mentioned integrability.

Application of these potentials is a separate and independent problem. These potentials are of practical importance, which approximate some real forces.

The author is grateful to professor A.Zhubr for useful comments and discussions.

Литература—References

1. *Aksenov E.P.* Teorija dvizhenija iskusstvennyh sputnikov zemli [Theory of the motion of the Earth’s artificial satellites]. Moscow: Nauka, 1977. 360 p. *Аксенов Е.П.* Теория движения искусственных спутников земли. М.: Наука, 1977. 360 с.
2. *Ferrandiz J.M., Floria L.* Towards a systematic definition of intermediaries in the theory of artificial satellites // Bull. Astron. Inst. Czechosl. 1991. Vol. 42. P. 401 – 407.
3. *Beletsky V.V.* Traektorii kosmicheskikh poletov s postojannym vektorom reaktivnogo uskorenija [Trajectories of Space Flights with a Constant Vector of Reactive Acceleration] // Cosmic Research. 1964. Vol. 2, No. 3. P. 787 – 807. *Белецкий В.В.* О траекториях космических

- полетов с постоянным вектором реактивного ускорения // Космические исследования. 1964. Т. 2, № 3. С. 787 – 807.
4. *Kunitsyn A.L.* O dvizhenii rakety v central'nom silovom pole s postojannym vektorom reaktivnogo uskorenija [Rocket Motion in a Central Force Field with a Constant Vector of Reactive Acceleration] // Cosmic Research. 1966. Vol. 4, No. 2. P. 324 – 332.
Куницын А.Л. О движении ракеты в центральном силовом поле с постоянным вектором реактивного ускорения // Космические исследования. 1966. Т. 4, № 2. С. 324 – 332.
 5. *Demin V.G.* Dvizhenie iskusstvennogo sputnika v nentral'nom pole tjagotenija [The motion of an artificial satellite in the eccentric gravitational field]. Moscow: Nauka, 1968. 352 p.
Демин В.Г. Движение искусственного спутника в нецентральной поле тяготения. М.: Наука, 1968. 352 с.
 6. *Kirchgraber U.* A problem of orbital dynamics, which is separable in KS -variables // Celest. Mech. 1971. Vol. 4. P. 340 – 347.
 7. *Poleshchikov S.M.* One integrable case of the perturbed two-body problem // Cosmic Res. Vol. 42, No. 4. P. 398 – 407.
 8. *Poleshchikov S.M., Kholopov A.A.* Teorija L -matric i reguljarizacija uravnenij dvizhenija v nebesnoj mehanike [Theory of L -matrices and regularization of motion equations in Celestial Mechanics]. Syktyvkar: Syktyvkar Forest Inst., 1999. 255 p.
Полещиков С.М., Холопов А.А. Теория L -матриц и регуляризация уравнений движения в небесной механике. Сыктывкар: СЛИ, 1999. 255 с.
 9. *Poleshchikov S.M.* Regularization of motion equations with L -transformation and numerical integration of the regular equations // Celest. Mech. and Dyn. Astr. 2003. Vol. 85, No. 4. P. 341 – 393.
 10. *Pars L.A.* A treatise on analytical dynamics. NY: Wiley, 1965. 636 p.
 11. *Kholshevnikov K.V.* Ob integriruемости v nebesnoj mehanike [On the integrability in celestial mechanics] // Analiticheskaja nebesnaja mehanika [Analytical celestial mechanics]. Kazan: Kazan University publ., 1990. P. 5 – 10.
Холшевников К.В. Об интегрируемости в небесной механике // Аналитическая небесная механика. Казань: Изд-во Казан. ун-та, 1990. С. 5 – 10.
 12. *Stiefel E., Scheifele G.* Linear and regular celestial mechanics. Berlin: Springer-Verlag, 1971. 304 p.
 13. *Poleshchikov S.M.* Regularization of canonical equations of the two-body problem using a generalized KS -matrix // Cosmic Res. 1999. Vol. 37, No. 3. P. 302 – 308.
 14. *Poleshchikov S.M.* Motion of a particle in a perturbed field of the attracting centre // Cosmic Res. 2007. Vol. 45, No. 6. P. 522 – 535.
 15. *Byrd P.F., Friedman M.D.* Handbook of elliptic integrals for engineers and physicists. Berlin: Springer-Verlag, 1954. 355 p.
 16. *Poleshchikov S.M., Zhubr A.V.* A set of potentials allowing integration of the perturbed two-body problem in regular coordinates // Cosmic Res. 2008. Vol. 46, No. 3. P. 202 – 214.
 17. *Poleshchikov S.M.* Integriruemyj sluchaj voz-mushhennoj zadachi dvuh tel, porozhdajushhej elementarnye funkicii [An integrable case of the perturbed two-body problem producing elementary functions] // Proc. of Syktyvkar Forest Inst. 2006. Vol. 6. P. 31 – 57.
Полещиков С.М. Интегрируемый случай возмущенной задачи двух тел, порождающей элементарные функции // Труды СЛИ. 2006. Т. 6. С. 31 – 57.

Статья поступила в редакцию 01.11.2021.