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РЕДУКЦИИ СИММЕТРИИ УРАВНЕНИЙ ЛИНДБЛАДА – ПРОСТЫЕ ПРИМЕРЫ И ПРИЛОЖЕНИЯ

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Аннотация

Открытая квантовая динамика в марковском приближении описывается основным уравнением Линдблада. Динамика Линдблада замкнута в алгебре Ли $\Lambda = su(n)$, т.е. имеет $su(n)$ симметрию. Мы говорим, что уравнение Линдблада допускает редукцию симметрии, если оно имеет инвариантное векторное подпространство $\Lambda_0 \subset \Lambda$ с Лиалгебраической структурой. Редукции симметрии ограничивают динамику на меньшие подпространства, которые дополнительно являются алгебрами Ли.

В заметке описаны тривиальные редукции, основанные на приводимости гамильтониана и операторов Линдблада. Представлены примеры нетривиальных редукций в пределе бесконечной температуры и редукций Майораны с сохранением четности. Обсуждаются приложения к открытой спиновой динамике.

Ключевые слова:
открытые квантовые системы, уравнение Линдблада, редукция симметрии

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SYMMETRY REDUCTIONS OF LINDBLAD EQUATIONS – SIMPLE EXAMPLES AND APPLICATIONS

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Abstract

Open quantum dynamics in the Markovian approximation is described by the Lindblad master equation. The Lindbladian dynamics is closed in the Lie algebra $\Lambda = su(n)$, i.e. it has $su(n)$ symmetry. We say that the Lindblad equation admits a symmetry reduction if it has an invariant vector subspace $\Lambda_0 \subset \Lambda$ with the Lie algebraic structure. Symmetry reductions restrict dynamics to smaller subspaces that additionally are Lie algebras.

In these notes, trivial reductions relying on the reducibility of the Hamiltonian and Lindblad operators are described. Examples of nontrivial reductions in the infinite temperature limit and the parity preserving Majorana reductions are presented. Applications to open spin dynamics are discussed.

Keywords:
open quantum systems, Lindblad master equation, symmetry reduction

Introduction

The open quantum dynamics in terms of the positive density operator in the Markovian approximation is described by the Lindblad master equation [1]

$$\dot{\rho} = \mathcal{M}\rho \equiv -i[H,\rho] + \mathcal{D}\rho,\ \mathcal{D} = \sum_{k=1}^{\bar{m}} \gamma_k \mathcal{L}(V_k),$$
$$\mathcal{L}(V)\rho \equiv V\rho V^\dagger - \frac{1}{2}\left(V^\dagger V\rho + \rho V^\dagger V\right). \tag{1}$$

Here H is the Hamiltonian, $\mathcal{D}$ is the dissipator built with the traceless Lindblad operators V_k, $\operatorname{Tr} V_k = 0$, and the non-negative rates $\gamma_k \geqslant 0$.

The density operator has trace 1 and at any time t is written in the form

$$\rho(t) = n^{-1}I + \rho_0(t), \quad \operatorname{Tr}\rho_0(t) = 0, \quad \rho_0(t) = \rho_0(t)^\dagger$$

where n is the dimension of the Hilbert space, I is the unit operator, $\rho_0(t)$ is the traceless Hermitian operator. The first term does not change in time. The vector space Λ of all possible traceless parts generate the Lie algebra (with the usual commutation of operators) that is isomorphic to $su(n)$, the algebra of traceless anti-Hermitian $n\times n$ operators. Indeed, multiplying traceless Hermitian operators by the complex unit, we come to traceless anti-Hermitian operators. Thus, the Lindbladian dynamics of Eq. (1) is closed in the Lie algebra $\Lambda = su(n)$, i.e., it has the $su(n)$ symmetry.

We say that Eq. (1) admits a symmetry reduction if it has a smaller invariant vector space $\Lambda_0 \subset \Lambda$ with a Lie algebraic structure. In other words, Λ_0 is a Lie algebra (with the usual commutator of operators) and any trajectory that starts in Λ_0 remains there for any times,

$$[\Lambda_0, \Lambda_0] \subset \Lambda_0, \quad \rho_0(0) \in \Lambda_0 \quad \longrightarrow \quad \forall t \quad \rho_0(t) \in \Lambda_0.$$

Obviously, Λ_0 is a subalgebra of the total symmetry algebra $su(n)$. Symmetry reductions restrict the dynamics into smaller invariant subspaces that additionally have a Lie algebraic structure.

Since all initial conditions within Λ_0 generate trajectories that stay within Λ_0 for all times, it is necessary that the action of the superoperator $\mathcal{M}$ to the unit operator belongs to Λ_0 and Λ_0 is invariant under the action of $\mathcal{M}$,

$$\mathcal{M}I \in \Lambda_0, \quad \mathcal{M}\Lambda_0 \subset \Lambda_0. \qquad (2)$$

Eq. (2) gives the criterion for the subalgebra Λ_0 to be a symmetry reduction.

In particular, the full Krylov subspace generated by the powers $\mathcal{M}^k I$ is within any symmetry reduction algebra Λ_0. The subspace K_I and so all symmetry reductions contain also the 1-dimensional subspace spanned by the equilibrium state (perhaps not unique),

$$\operatorname{span}\{\rho_0^*\} \in \Lambda_0, \quad n^{-1}\mathcal{M}I + \mathcal{M}\rho_0^* = 0. \qquad (3)$$

Thus, for existence of symmetry reductions it is necessary that the subspace K_I is a proper subspace of the total algebra $su(n)$,

$$\begin{gathered} \dim K_I < n^2 - 1, \\ K_I = \operatorname{span}\{\mathcal{M}^k I,\ k = 1,\, 2,\, \ldots\}. \end{gathered} \qquad (4)$$

Indeed, the Krylov subspace K_I is an invariant subspace of Eq. (1), i.e., trajectories starting in K_I remain there all the time. In general, $\dim K_I = n^2 - 1$, the Krylov subspace coincides with the total symmetry algebra $su(n)$, Eq. (1) does not have proper invariant subspaces and hence does not admit symmetry reductions.

Eqs. (2), (3), (4) show that the Hamiltonian and Lindblad operators should satisfy special conditions for Eq. (1) to have symmetry reductions. In these notes, we discuss first trivial symmetry reductions relying on splitting the Hilbert space by reducibility of the Hamiltonian and Lindblad operators. Then we present two examples of nontrivial symmetry reductions: the reduction to the infinite temperature limit $\mathcal{M}I = 0$ and the reduction by the parity Z_2-grading of the total algebra $su(n)$ realised as a Majorana reduction. In the first example, the symmetry reduction is due to a constraint to the dissipation rates γ_k. The second example is valid for any dissipation rates. The relevant applications to open spin dynamics are pointed out and briefly discussed.

1. Trivial reduction

The Hamiltonian is an Hermitian operator. The set of Lindblad operators (even in useful physical models) is typically not very large and is subdivided into a set of Hermitian operators and a set of Hermitian-conjugate pairs,

$$\begin{gathered} V_{2q} = V_{2q-1}^\dagger \neq V_{2q-1}, \quad q = 1,\, \ldots,\, m_1, \\ H = H^\dagger, \quad V_{2m_1+p} = V_{2m_1+p}^\dagger, \quad p = 1,\, \ldots,\, m_2. \end{gathered}$$

We have then

$$\begin{aligned} \mathcal{D} = -\frac{1}{2}\sum_{p=1}^{m_2} \gamma_{2m_1+p}[V_{2m_1+p}, [V_{2m_1+p}, \cdot]] + \\ + \sum_{q=1}^{m_1} \left\{ \gamma_{2q}\mathcal{L}(V_{2q}) + \gamma_{2q-1}\mathcal{L}(V_{2q}^\dagger) \right\}. \end{aligned} \qquad (5)$$

In particular,

$$\mathcal{M}I = \sum_{q=1}^{m_1} (\gamma_{2q} - \gamma_{2q-1})[V_{2q}, V_{2q}^\dagger]. \qquad (6)$$

Let the set of the Hermitian-conjugate pairs of Lindblad operators V_{2q}, $V_{2q}^\dagger$ be reducible, i.e., possesses a common invariant vector subspace $X \subset h$ of the Hilbert space of a lower dimension,

$$\begin{gathered} V_{2q}X \subset X, \quad V_{2q}^\dagger X \subset X, \quad q = 1,\, \ldots,\, m_1, \\ 0 < \dim X = m < n. \end{gathered}$$

Then the Hermitian-conjugate part of the dissipator (given by the second term in Eq. (5)) is closed in the set Λ_0 of all traceless operators $\rho_0 \in su(n)$ that preserve the reduced subspace X,

$$\rho \in \Lambda_0 \quad \longrightarrow \quad \rho_0 X \subset X.$$

The set Λ_0 is closed with respect to commutation of operators and so forms a Lie algebra. This algebra is isomorphic to the $su(n)$-normalizer of the algebra $su(m)$ spanned by traceless operators on the reduced subspace X. If additionally the Hamiltonian and the Hermitian Lindblad operators belong to Λ_0,

$$V_{2m_1+p},\, H \in \Lambda_0 = N(su(m)),$$

then the subalgebra Λ_0 is a symmetry reduction of Eq. (1). Since the complementary subspace $X_c = h\backslash X$, $\dim X_c = n - m$ is also invariant for V_{2q}, $V_{2q}^\dagger$, the same construction is applicable to X_c. Hence,

$$\Lambda_0 = N(su(m)) = N(su(n-m))$$

where the subalgebra $su(n-m)$ is spanned by traceless operators on the complementary subspace X_c.

We call such symmetry reduction a *trivial reduction*, as it describes the situation where the initial Hilbert space, being formally n-dimensional, is in fact split into two independent subspaces of lower dimensions m, $n - m < n$ that are not dynamically connected. We have (after a suitable permutation of Hilbert states)

$$\begin{gathered} H, \quad V_k \in \Lambda_0 = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A}_c \end{pmatrix}, \\ \dim \mathcal{A} = m \times m, \quad 0 < m < n. \end{gathered}$$

In the special case where the Lindblad operators are all Hermitian, $V_{2q} = 0$, $q = 1, \ldots, m_1$, the symmetry reductions are any subalgebras which normalizers contain the operators V_{2m_1+p}, H.

2. Infinite temperature reduction

Let the unit operator annihilate the right-hand side of Eq. (1),

$$\mathcal{M}I = 0 \tag{7}$$

and the operators $[V_{2q}, V_{2q}^\dagger]$ are linearly independent. Then it follows from Eq. (6) that

$$\gamma_{2q} = \gamma_{2q-1} \equiv \bar{\gamma}_q. \tag{8}$$

As a result, according to Eq. (5),

$$\begin{aligned} \mathcal{D} = &-\frac{1}{2}\sum_{p=1}^{m_2} \gamma_{2m_1+p}[V_{2m_1+p}, [V_{2m_1+p}, \cdot]]- \\ &-\frac{1}{2}\sum_{q=1}^{m_1} \bar{\gamma}_q \left([V_{2q}, [V_{2q}^\dagger, \cdot]] + [V_{2q}^\dagger, [V_{2q}, \cdot]]\right). \end{aligned} \tag{9}$$

Let Λ_0 be the subalgebra generated by the Hermitian-conjugate pairs of Lindblad operators and let the Hermitian Lindblad operators and the Hamiltonian belong to the normalizer of Λ_0,

$$[V_{2m_1+p}, \Lambda_0],\ [H, \Lambda_0] \subset \Lambda_0 \ \longleftarrow\ \{V_{2q},\ V_{2q}^\dagger\}. \tag{10}$$

It follows then from Eq. (9) that Λ_0 is a symmetry reduction of Eq. (1).

According to Eq. (7), the density operator $\rho = n^{-1}I$ is an incoherent equilibrium solution to Eq. (1), featuring the case where all the pure states of the system have equal probabilities. This case corresponds to the infinite temperature limit of both Fermi-Dirac and Bose-Einstein statistics that in this case coincide with the corresponding limit of the Boltzmann distribution. For this reason, we call the symmetry reduction given by Eqs. (7), (8), (10) an *infinite temperature reduction*.

As an example, consider the Markovian Lindblad open dynamics of a single spin in a magnetic field. The Hamiltonian contains the Zeeman splitting and the coherent driving parts,

$$H = \Omega S_z + \frac{\omega_1}{2}\left(S_+ + S_-\right).$$

The Lindblad operators are the usual raising and lowering operators characterizing the longitudinal relaxation plus the z-operator that describes the transverse relaxation,

$$V_{1,2} = S_\pm = S_x \pm iS_y, \quad V_3 = S_z.$$

We assume that the initial value for the dynamics belongs to the spin algebra $so(3)$ (treated as an isomorphic copy of $su(2)$),

$$\begin{aligned} &\rho(0) = n^{-1}I + p_z S_z + p_+ S_+ + p_- S_-, \\ &[S_z, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_z \end{aligned} \tag{11}$$

where $n = 2s+1$ is the dimension of the Hilbert space, s is the spin quantum number (any half-integer or integer), $p_{z,\pm}$ are some constants.

Let $\gamma_1 = \gamma_2 \equiv \gamma$, i.e., the Lindblad operators $S_\pm$ have the same dissipation rates. Then it can be easily verified that we are under conditions of the infinite temperature reduction described earlier in this section. For any time t the dynamics remains closed in $so(3)$, i.e., is reduced to a 3-dimensional dynamics of the constants $p_{z,\pm}$,

$$\rho(t) = n^{-1}I + p_z(t)S_z + p_+(t)S_+ + p_-(t)S_-.$$

The latter is described by the well-known Bloch equations. This result is valid for any spin quantum number s.

Let now $s > 1/2$ and $\gamma_1 \neq \gamma_2$. Then Eq. (7) no longer holds and the dynamics, even starting in $so(3)$ as in Eq. (11), at $t > 0$ comes out of the spin algebra $so(3)$. Indeed, the (non-driven) thermal equilibrium state is represented now by a diagonal matrix that cannot be represented by a combination of the unit operator and the operator S_z,

$$\begin{aligned} \rho_{th} = Z^{-1}\exp\left(-\frac{\hbar\bar{\Omega}S_z}{kT}\right) = \\ = \mathrm{diag}\{\rho_1, \ldots, \rho_n\} \neq n^{-1}I + \beta S_z. \end{aligned}$$

As a matter of fact, higher orders S_z^r, $r > 1$, of the operator S_z occur that do not belong to $so(3)$. This corresponds to the case of finite temperatures.

It can be shown that in the presence of the coherent terms in the Hamiltonian, $\omega_1 \neq 0$, and for $\gamma_1 \neq \gamma_2$, the above spin dynamics does not admit symmetry reductions: the dimension of any (non-equilibrium) trajectory equals $n^2 - 1$, the dimension of the total symmetry algebra $su(n)$. For $\omega_1 = 0$ and any $\gamma_{1,2}$ the dynamics admits the symmetry reduction represented by the algebra of traceless diagonal $n \times n$ matrices that is the Cartan subalgebra of $su(n)$.

3. Parity preserving reduction

The total symmetry algebra $\Lambda = su(n)$ is a formal superalgebra, i.e., it admits a Z_2-grading into even and odd subspaces respected by the commutation,

$$\Lambda = \Lambda_0 + \Lambda_1, \quad [\Lambda_l, \Lambda_j] \subset \Lambda_{(l+j)(\mathrm{mod}\,2)}.$$

Here the even subspace Λ_0 is a subalgebra, while the odd subspace Λ_1 (being not a subalgebra) complements the even subalgebra to the total algebra Λ.

It can be assumed that the Z_2-grading of $su(n)$ is inherited from a Z_2-grading of the associative algebra of $n \times n$ matrices. An example valid for any n is the representation of traceless (anti-)Hermitian operators by matrix elements ρ_{lj}, $1 \leqslant l, j \leqslant n$, that belong to even and odd collateral diagonals,

$$\Lambda_s = \mathrm{span}\,\{\rho_{lj} :\ l - j = s\,(\mathrm{mod}\,2)\},\ s = 0, 1. \tag{12}$$

The following result is a simple subsequence of the above construction. If the Hamiltonian belongs to the even subspace, while each Lindblad operator belongs entirely to one of the two grading subspaces,

$$H \in \Lambda_0, \quad V_k \in \Lambda_0 \text{ or } \Lambda_1,$$

then the even subalgebra Λ_0 is a symmetry reduction of Eq. (1). We call such symmetry reduction a *parity preserving reduction*.

For large n, the Z_2-grading given by Eq. (12) may be not unique. Below we consider an example of parity preserving reduction based on the so-called Majorana fermions.

The Majorana fermions form a set of $2m$ $n\times n$ operators a_l, $l=1,\ldots,2m$, that anti-commute with each other and square to the unit operator,

$$a_l^2 = I, \quad a_l a_j + a_j a_l = 0, \quad l \neq j.$$

Consider the even and odd parts of the Clifford algebra generated by the Majorana fermions, i.e., the vector spaces spanned by even and odd order products of the operators a_l,

$$\Lambda_0 = \operatorname{span}\Big\{ \prod_{r=1}^{2k} a_{l_r},\ l_r = 1,\ldots,2m,\quad k = 1,\ldots,m \Big\},$$

$$\Lambda_1 = \operatorname{span}\Big\{ \prod_{r=1}^{2k-1} a_{l_r},\ l_r = 1,\ldots,2m,\quad k = 1,\ldots,m \Big\}.$$

Denote Λ_0, Λ_1 the subspaces of the above subspaces that contain only traceless operators. The even subspace Λ_0 is closed with respect to the operator commutation and so forms a Lie algebra. It follows from the anti-commutation of the Majorana fermions that the algebra Λ_0 contains the algebra of anti-symmetric quadratic forms of the operators a_l that is isomorphic to the algebra $so(2m)$ of (generally complex) anti-symmetric operators. Hence, Λ_0 is an extension of $so(2m)$ by all even order products of a_l.

The above construction suggests that the subdivision $\Lambda = \Lambda_0 + \Lambda_1$ is a Z_2-grading of the total symmetry algebra. Thus, if the Hamiltonian belongs to Λ_0 and each Lindblad operator belongs to either Λ_0 or Λ_1, then the algebra Λ_0 is a symmetry reduction of Eq. (1). In other words, for any time t the traceless part of the density operator is represented by a combination of even order products of the operators a_l. We call this a *Majorana reduction*.

As an example, consider the system of $m > 1$ interacting spin-1/2 particles (for example, qubits). This system is described by the tensor product of m 4-dimensional operator spaces spanned by the set of 2×2 operators $1,\ S_x,\ S_y,\ S_z$ where 1 is the unit operator and S_α, $\alpha = x, y, z$, are the standard spin-1/2 operators (of the 2-representation of $so(3)$). As each spin has two states and the trace of the density operator equals 1, the whole symmetry algebra is $\Lambda = su(2^m)$ of dimension $4^m - 1$.

Denote $S_{kx},\ S_{ky},\ S_{kz}$ the spin operators generated by the kth spin – that placed in position k of the tensor product: $S_{k\alpha} = 1\times\ldots\times 1\times S_\alpha\times 1\ldots\times 1$. These individual operators for different spins $k \neq k'$ commute and generate the algebra $so(3)$ within their subspaces with the same k. It can be easily verified that the set of $2m$ operators

$$\begin{aligned} a_{2k-1} &= 2^k S_{kx}\prod_{s<k} S_{sz},\\ a_{2k} &= 2^k S_{ky}\prod_{s<k} S_{sz},\\ k &= 1,\ldots,m, \end{aligned} \tag{13}$$

are Majorana fermions.

Here, the Majorana chain

$$H = \sum_{l=1}^{2m-1} h_l[a_l, a_{l+1}] = \sum_{l=1}^{m-1} (\Omega_l S_{lz} + D_l S_{lx} S_{l+1,x})$$

leads to the Hamiltonian of the coherently driven 1-dimensional Ising model (with respect to the x axis). Another combination

$$H = \sum_{l=1}^{2m-1} h_l[a_l, a_{l+1}] + \sum_{k=1}^{m-1} \bar{h}_k[a_{2k-1}, a_{2k+2}] = \sum_{l=1}^{m-1}\left(\Omega_l S_{lz} + D_l^x S_{lx} S_{l+1,x} + D_l^y S_{ly} S_{l+1,y}\right)$$

leads to the Hamiltonian of the driven anisotropic Heisenberg XY 1-dimensional chain. Adding higher even order Majorana products, we can get interacting spin systems of higher dimensions. In all cases, choosing various combinations of odd or even order Majorana products as the Lindblad operators, we generate various dissipation models.

Other physically important examples of symmetry reductions relying on parity can be suggested.

References

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