Stückelberg particle in external magnetic field. The method of projective operators

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Abstract

We study the Stückelberg equation for a relativistic particle with two spin states $S = 1$ and $S = 0$ in the presence of an external uniform magnetic field. The particle is described by an 11-component wave function consisting of a scalar, a vector, and an antisymmetric tensor. On the solutions of the equation, the operators of energy, the third projection of the total angular momentum, and the third projection of the linear momentum along the direction of the magnetic field are diagonalized. After separation of variables, a system for 11 radial functions is obtained. Its solution is based on the use of the Fedorov-Gronsky method, in which all 11 radial functions are expressed in terms of three main functions. Exact solutions with cylindrical symmetry are constructed. Three series of energy levels are found.

Keywords:

Stückelberg particle, magnetic field, projective operators, Fedorov–Gronskiy method, exact solutions, bound states

Частица Штюкельберга во внешнем магнитном поле. Метод проективных операторов

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Аннотация

Исследуется уравнение Штюкельберга для релятивистской частицы с двумя спиновыми состояниями $S = 1$ и $S=0$ в присутствии внешнего однородного магнитного поля. Частица описывается 11-компонентной волновой функцией, состоящей из скаляра, вектора и антисимметричного тензора. На решениях уравнения диагонализируются операторы энергии, третьей проекции полного углового момента и третьей проекции линейного момента вдоль направления магнитного поля. После разделения переменных получена система для 11 радиальных функций. Ее решение основано на использовании метода Федорова-Гронского, в рамках которого все 11 радиальных функций выражаются через три основные функции. Построены точные решения с цилиндрической симметрией. Найдены три серии уровней энергии.

Ключевые слова:

частица Штюкельберга, магнитное поле, проективные операторы, метод Федорова–Гронского, точные решения, связанные состояния

Introduction

In the literature, the great interest can be noticed in studying the Dirac-Kähler field [1,2]; also see [3–10]. This field describes the complicated boson which contains the fields with different parities and spins $S = 1, S = 0$; the complete wave function includes the scalar, pseudoscalar, vector, pseudovector, and antisymmetric tensor components (pseudo-quantities are marked by symbol of tilde): $\Phi, \tilde{\Phi}, \Phi_a, \tilde{\Phi}_a, \Phi_{ab}$. In the frames of the general theory of relativistic wave equations [16–20], the Dirac-Kähler field describes a particle with a set of spin states: $S = 1$ and $S = 0$ (see [11–15]).

From this theory, by imposing additional constraints on the 16 components of the Dirac-Kähler field, we can obtain the usual theories for scalar and pseudoscalar particles, and for vector and pseudovector particles:

$$(\Phi, 0, \Phi_a, 0, 0), \quad S = 0;$$

$$(0, \tilde{\Phi}, 0, \tilde{\Phi}_a, 0), \quad S = \tilde{0};$$

$$(0, 0, \Phi_a, 0, \Phi_{ab}), \quad S = 1;$$
$$(0, 0, 0, \tilde{\Phi}_a, \Phi_{ab}), \quad S = \tilde{1}.$$

Also the system of equations describing the Stückelberg particle is well known [1, 2]. In particular, this system can be obtained from the Dirac-Kähler theory by imposing the following constraints

$$(\Phi, 0, \Phi_a, 0, \Phi_{ab}), \quad S = 0, 1;$$
$$(0, \tilde{\Phi}, 0, \tilde{\Phi}_a, \Phi_{ab}), \quad S = \tilde{0}, \tilde{1}.$$

There are possible two theories, corresponding to different internal parities of the particle. In this paper the first variant of the Stückelberg theory is studied.

We start with the known Stückelberg tensor system of 11 equations, which is transformed to a matrix form with the use of the tetrad method [11, 12]. This equation is detailed in cylindric system of coordinates and corresponding tetrad; wherein we take into account the external uniform magnetic field. We perform the separation of the variables and derive the system of 11 equations in r variable. To resolve this system we apply the method of Fedorov-Gronskiy [21], which is based on projective operators referring to the third projection of the 11-dimensional spin matrix. According to this method we can express all 11 functions through only three ones. We find 3 series of physically interpretable energy levels, as solutions of algebraic equations of order 1 and 3.

1. The basic equation

The initial Stückelberg system of equations is the following

$$-D^a \Psi_a - \mu \Psi = 0,$$
$$D_a \Psi + D^b \Psi_{ab} - \mu \Psi_a = 0,$$
$$D_a \Psi_b - D_b \Psi_a - \mu \Psi_{ab} = 0,$$

where $D_a = \partial_a + ieA_a$. The relation of the parameter μ to physical mass of the particle M is given by $\mu = -M$. We use the 11-dimensional wave function in the form

$$\Phi = (\Psi; \Psi_0, \Psi_1, \Psi_2, \Psi_3; \Psi_{01}, \Psi_{02},$$
$$\Psi_{03}, \Psi_{23}, \Psi_{31}, \Psi_{12}) = (H, H_1, H_2).$$

The above system can be presented in the matrix block form

$$D_a G^a H_1 + \mu H = 0,$$
$$\Delta^a D_a H + K^a D_a H_2 - \mu H_1 = 0,$$
$$D_a L^a H_1 - \mu H_2 = 0,$$

or (note the minus sign in front of G^a)

$$(-D_a \Gamma^a - \mu)\Phi = 0,$$

$$\Gamma^a = \begin{pmatrix} 0 & -G^a & 0 \\ \Delta^a & 0 & K^a \\ 0 & L^a & 0 \end{pmatrix}, \Phi = \begin{pmatrix} H \\ H_1 \\ H_2 \end{pmatrix}, \qquad (1)$$

where

$$\Delta^0 = (1, 0, 0, 0)^t, \quad \Delta^1 = (0, 1, 0, 0)^t,$$

$$\Delta^2 = (0, 0, 1, 0)^t, \quad \Delta^3 = (0, 0, 0, 1)^t,$$

$$K^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$K^1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$K^2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$K^3 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L^1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$L^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$L^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here and below, t stands for transposition.

This matrix Stückelberg equation can be extended to the Riemannian space-time in accordance with the known procedure. To this end, for any given metric $g_{\alpha\beta}(x)$ we should take the certain tetrad:

$$g_{\alpha\beta}(x) \to e_{(a)\alpha}(x), \quad \eta^{ab} = diag(1, -1, -1, -1),$$

then the equation (1) should have the structure

$$\left[\Gamma^\alpha(x) \left(\frac{\partial}{\partial x^\alpha} + \Sigma_\alpha(x) \right) - \mu \right] \Psi(x) = 0. \qquad (2)$$

Local matrices $\Gamma^{\alpha}(x)$ and their blocks are determined with the use of the tetrad

$$\Gamma^{\alpha}(x) = e^{\alpha}_{a}(x)\Gamma^{a} = \begin{pmatrix} 0 & -G^{a}e^{\alpha}_{(a)} & 0 \\ \Delta^{a}e^{\alpha}_{(a)} & 0 & K^{a}e^{\alpha}_{(a)} \\ 0 & L^{a}e^{\alpha}_{(a)} & 0 \end{pmatrix}.$$

The connection $\Sigma_{\alpha}(x)$ is defined by the formulas

$$J^{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J^{ab}_{1} & 0 \\ 0 & 0 & J^{ab}_{2} \end{pmatrix},$$

$$\Sigma_{\alpha}(x) = \frac{1}{2}J^{ab}e^{\beta}_{(a)}(x)e_{(b)\beta;\alpha}(x) =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\Sigma_{1})_{\alpha} & 0 \\ 0 & 0 & (\Sigma_{2})_{\alpha} \end{pmatrix},$$

$$\Sigma_{1}(x) = \frac{1}{2}J^{ab}_{(1)}e^{\beta}_{(a)}(x)e_{(b)\beta;\alpha}(x),$$

$$\Sigma_{2}(x) = \frac{1}{2}J^{ab}_{(2)}e^{\beta}_{(a)}(x)e_{(b)\beta;\alpha}(x),$$

where $J^{ab}_{(1)}$ and $J^{ab}_{(2)}$ designate generators for vector $\Psi_{k}(x)$ and antisymmetric tensor $\Psi_{[mn]}(x)$, respectively. Equation (2) may be presented with the use of the Ricci rotation coefficients

$$\left[\Gamma^{c}\left(e^{\alpha}_{(c)}\frac{\partial}{\partial x^{\alpha}} + \frac{1}{2}J^{ab}\gamma_{abc}\right) - \mu\right]\Psi(x) = 0. \quad (3)$$

Recall that $\gamma_{[ab]c} = -\gamma_{[ba]c} = e_{(b)\rho\sigma}e^{\rho}_{(a)}e^{\sigma}_{(c)}$. In detailed form, Eq. (3) reads

$$\left[-G^{c}e^{\alpha}_{(c)}\partial_{\alpha} - G^{c}J^{ab}_{(1)}\frac{1}{2}\gamma_{abc}\right]H_{1} - \mu H = 0,$$

$$\Delta^{c}e^{\alpha}_{(c)}\partial_{\alpha}H + \left[K^{c}e^{\alpha}_{(c)}\partial_{\alpha} + K^{c}J^{ab}_{(2)}\frac{1}{2}\gamma_{abc}\right]H_{2} - \mu H_{1} = 0,$$

$$\left[L^{c}e^{\alpha}_{(c)}\partial_{\alpha} + L^{c}J^{ab}_{(1)}\frac{1}{2}\gamma_{abc}\right]H_{1} - \mu H_{2} = 0.$$

Let us consider the Stückelberg equation in presence of the external uniform magnetic field. In cylindrical coordinates with the use of the diagonal tetrad

$$x^{\alpha} = (t, r, \phi, z), \quad dS^{2} = dt^{2} - dr^{2} - r^{2}d\phi^{2} - dz^{2},$$

$$A_{\phi} = -\frac{Br^{2}}{2}$$

the above equation takes the form (let $eB \Rightarrow B$):

$$\left[\Gamma^{0}\frac{\partial}{\partial t} + \Gamma^{1}\frac{\partial}{\partial r} + \Gamma^{2}\frac{\partial_{\phi} + iBr^{2}/2 + J^{12}}{r} + \right.$$
$$\left. + \Gamma^{3}\frac{\partial}{\partial z} - \mu\right]\Psi = 0.$$

In block form, it reads

$$\left[-G^{0}\frac{\partial}{\partial t} - G^{1}\frac{\partial}{\partial r} - G^{2}\frac{1}{r}\left(\frac{\partial}{\partial \phi} + \frac{iBr^{2}}{2} + j^{12}_{1}\right) - \right.$$
$$\left. - G^{3}\frac{\partial}{\partial z}\right]H_{1} - \mu H = 0,$$

$$\left[\Delta^{0}\frac{\partial}{\partial t} + \Delta^{1}\frac{\partial}{\partial r} + \right.$$
$$\left. + \Delta^{2}\frac{1}{r}\left(\partial\phi + \frac{iBr^{2}}{2}\right) + \Delta^{3}\frac{\partial}{\partial z}\right]H +$$
$$+ \left[K^{0}\frac{\partial}{\partial t} + K^{1}\frac{\partial}{\partial r} + K^{2}\frac{\partial\phi + \frac{iBr^{2}}{2} + j^{12}_{1}}{r} + \right.$$
$$\left. + K^{3}\frac{\partial}{\partial z}\right]H_{2} = \mu H_{1},$$

$$\left[L^{0}\frac{\partial}{\partial t} + L^{1}\frac{\partial}{\partial r} + L^{2}\frac{\partial_{\phi} + \frac{iBr^{2}}{2} + j^{12}_{1}}{r} + \right.$$
$$\left. + L^{3}\frac{\partial}{\partial z}\right]H_{1} = \mu H_{2}.$$

2. Cyclic basis

In the following, it will be convenient to apply the cyclic basis (all quantities referring to it are marked by the overline). In such a basis, the generators j^{12}_{1} and j^{12}_{2} are diagonal. The necessary transformation $\bar{H}_{1} = UH_{1}$ is determined by the matrix U :

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix},$$

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \\ 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & \end{pmatrix}.$$

Correspondingly, the generators for tensor representation are defined by the rule

$$\bar{J}^{ab}_{1} = Uj^{ab}U^{-1}, \quad \bar{J}^{ab}_{2} = \bar{j}^{ab}\otimes I + I\otimes\bar{j}^{ab}.$$

Let us transform the generators to cyclic form

$$\bar{j}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$\bar{J}^{12}_{2} = \begin{pmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{pmatrix}.$$

We should also transform the main matrices Γ^a of the equation to the cyclic form. Starting with the formulas

$$\bar{H} = H, \quad \bar{H}_1 = C_1 H_1, \quad (C_1 = U),$$
$$\bar{H}_2 = U \otimes U H_2 = C_2 H_2,$$

we derive the rule

$$\begin{pmatrix} 0 & -\bar{G}^a & 0 \\ \bar{\Delta}^a & 0 & \bar{K}^a \\ 0 & \bar{L}^a & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & -G^a C_1^{-1} & 0 \\ C_1 \Delta^a & 0 & C_1 K^a C_2^{-1} \\ 0 & C_2 L^a C_1^{-1} & 0 \end{pmatrix}.$$

First, we find the matrices $\bar{\Delta}^a = C_1 \Delta^a$ and $\bar{G}^a = G^a C_1^{-1}$:

$$\bar{\Delta}^0 = (1,0,0,0)^t, \quad \bar{\Delta}^1 = \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^t$$

$$\bar{\Delta}^2 = \left(0, \frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}}\right)^t, \quad \bar{\Delta}^3 = (0,0,1,0)^t;$$

$$\bar{G}^0 = (1,0,0,0), \quad \bar{G}^1 = \left(0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right),$$

$$\bar{G}^2 = \left(0, \frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}}\right), \quad \bar{G}^3 = (0,0,-1,0).$$

Having in mind the formula for C_1, we can derive expressions for 6-dimensional transformation for C_2:

$$U \otimes U \Rightarrow C_2, \ \bar{H}_2 = C_2 H_2 \Rightarrow$$

$$C_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$C_2^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix}.$$

With the use of them we can obtain all other blocks in cyclic form:

$$\bar{K}^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{K}^1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$\bar{K}^2 = \begin{pmatrix} \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix},$$

$$\bar{K}^3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix};$$

$$\bar{L}^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{L}^1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$\bar{L}^2 = \begin{pmatrix} -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix},$$

$$\bar{L}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

3. Separating the variables

We apply the following substitution for the wave function (in cyclic basis)

$$\bar{\Psi} = e^{-i\epsilon t} e^{im\phi} e^{ikz} \begin{pmatrix} \bar{H} \\ \bar{H}_1 \\ \bar{H}_2 \end{pmatrix}, \ \bar{H} = h(r),$$

$$\bar{H}_1 = \begin{pmatrix} h_0(r) \\ h_1(r) \\ h_2(r) \\ h_3(r) \end{pmatrix}, \ \bar{H}_2 = \begin{pmatrix} E_i(r) \\ B_i(r) \end{pmatrix}.$$

After a simple calculation we derive the system of 11 equations. With the use of notations

$$a_m = \frac{d}{dr} + \frac{m + Br^2/2}{r}, \ a_{m+1} = \frac{d}{dr} + \frac{m+1+Br^2/2}{r},$$

$$b_m = \frac{d}{dr} - \frac{m + Br^2/2}{r}, \ b_{m-1} = \frac{d}{dr} - \frac{m-1+Br^2/2}{r},$$

it reads

$$-i\epsilon h_0 - ikh_2 + b_{m-1}h_1 - a_{m+1}h_3 = -\mu h,$$

$$-i\epsilon h - ikE_2 + b_{m-1}E_1 - a_{m+1}E_3 = \mu h_0,$$

$$-a_m h + a_{m+1}B_2 - ikB_3 + i\epsilon E_1 = \mu h_1,$$

$$ikh + i\epsilon E_2 - a_{m+1}B_1 - b_{m-1}B_3 = \mu h_2,$$

$$b_m h + b_m B_2 + ikB_1 + i\epsilon E_3 = \mu h_3,$$

$$a_m h_0 - i\epsilon h_1 = \mu E_1, \quad -ikh_0 - i\epsilon h_2 = \mu E_2,$$

$$-b_m h_0 - i\epsilon h_3 = \mu E_3, \quad -b_m h_2 + ikh_3 = \mu B_1,$$

$$b_{m-1}h_1 + a_{m+1}h_3 = \mu B_2, \ -ikh_1 - a_m h_2 = \mu B_3.$$

4. The Fedorov-Gronskiy method

We will apply the Fedorov-Gronskiy method [21]. To this end, let us consider the third projection of 11-dimensional spin operator $Y = -i\bar{J}^{12}$. We verify that it satisfies the minimal cubic equation, $Y(Y-1)(Y+1) = 0$. This permits us to introduce three projective operators

$$P_1 = \frac{1}{2}Y(Y-1), \ P_2 = \frac{1}{2}Y(Y+1), \ P_3 = 1 - Y^2$$

with the properties $P_0^2 = P_0$, $P_{+1}^2 = P_{+1}$, $P_{-1}^2 = P_{-1}$, $P_0 + P_{+1} + P_{-1} = 1$.

Therefore, the complete wave function may be decomposed into the sum of three parts

$$\Psi = \Psi_0 + \Psi_{+1} + \Psi_{-1},$$

$$\Psi_\sigma = P_\sigma \Psi, \quad \sigma = 0, +1, -1.$$

We can readily find an explicit form of them (according to the Fedorov-Gronskiy method, each projective part should be determined by only one function)

$$\Psi_1(r) = (0, 0, h_1, 0, 0, E_1, 0, 0, 0, 0, B_3)^t f_1(r),$$

$$\Psi_2(r) = (0, 0, 0, 0, h_3, 0, 0, E_3, B_1, 0, 0)^t f_2(r),$$

$$\Psi_3(r) = (h, h_0, 0, h_2, 0, 0, E_2, 0, 0, B_2, 0)^t f_3(r).$$

Applying the projective operators to the system of 11 equations, $P_i(A_{10\times 10}\Psi) = 0$, we obtain

for P_1

$$-a_m h + a_m B_2 - ikB_3 + i\epsilon E_1 = \mu h_1,$$

$$a_m h_0 - i\epsilon h_1 = \mu E_1,$$

$$-ikh_1 - a_m h_2 = \mu B_3;$$

for P_2

$$b_m h + b_m B_2 + ikB_1 + i\epsilon E_3 = \mu h_3,$$

$$-b_m h_0 - i\epsilon h_3 = \mu E_3,$$

$$-b_m h_2 + ikh_3 = \mu B_1;$$

for P_3

$$-i\epsilon h_0 - ikh_2 + b_{m-1}h_1 - a_{m+1}h_3 = \mu h,$$

$$-i\epsilon h - ikE_2 + b_{m-1}E_1 - a_{m+1}E_3 = \mu h_0,$$

$$ikh + i\epsilon E_2 - a_{m+1}B_1 - b_{m-1}B_3 = \mu h_2,$$

$$-ikh_0 - i\epsilon h_2 = \mu E_2,$$

$$b_{m-1}h_1 + a_{m+1}h_3 = \mu B_2.$$

Besides, in accordance with the Fedorov-Gronskiy method, we impose the first order constraints which permit us to transform all differential equations into algebraic ones:

for P_1

$$- a_m f_3(r)h + a_m f_3(r)B_2 - ikf_1(r)B_3 + \\ + i\epsilon f_1(r)E_1 = \mu f_1(r)h_1 \Rightarrow a_m f_3 = C_1 f_1,$$

$$a_m f_3(r)h_0 - i\epsilon f_1(r)h_i = \\ = \mu f_1(r)E_1 \Rightarrow a_m f_3 = C_1 f_1,$$

$$- ikf_1(r)h_1 - a_m f_3(r)h_2 = \\ = \mu f_1(r)B_3 \Rightarrow a_m f_3 = C_1 f_1,$$

for P_2

$$b_m f_3(r)h + b_m f_3(r)B_2 + ikf_2(r)B_1 + \\ + i\epsilon f_2(r)E_3 = \mu f_2(r)h_3 \Rightarrow b_m f_3 = C_2 f_2,$$

$$- b_m f_3(r)h_0 - i\epsilon f_2(r)h_3 = \\ = \mu f_2(r)E_3 \Rightarrow b_m f_3 = C_2 f_2,$$

$$- b_m f_3(r)h_2 + ikf_2(r)h_3 = \\ = \mu f_2(r)B_1 \Rightarrow b_m f_3 = C_2 f_2,$$

for P_3

$$- i\epsilon f_3(r)h_0 - ikf_3(r)h_2 + b_{m-1}f_1(r)h_1 - \\ - b_{m-1}f_1(r)h_3 = \mu f_3(r)h \Rightarrow b_{m-1}f_1 = C_3 f_3,$$

$$- i\epsilon f_3(r)h - ikf_3(r)E_2 + b_{m-1}f_1(r)E_1 - \\ - a_{m+1}f_2(r)E_3 = \mu f_3(r)h_0 \Rightarrow \\ \Rightarrow b_{m-1}f_1 = C_3 f_3, \quad a_{m+1}f_2 = C_4 f_3,$$

$$ikf_3(r)h = i\epsilon f_3(r)E_2 - a_{m+1}f_2(r)B_1 - \\ - b_{m-1}f_1(r)B_3 = \mu f_3(r)h_2 \Rightarrow \\ \Rightarrow b_{m-1}f_1 = C_3 f_3, \quad a_{m+1}f_2 = C_4 f_3,$$

$$- ikf_3(r)h_0 - i\epsilon f_3(r)h_2 = \mu f_3(r)E_2, \\ b_{m-1}f_1(r)h_1 + a_{m+1}f_2(r)h_3 = \mu f_3(r)B_2 \Rightarrow \\ \Rightarrow b_{m-1}f_1 = C_3 f_3, \quad a_{m+1}f_2 = C_4 f_3.$$

Thus, we have derived the algebraic equations

$$-C_1 h + C_1 B_2 - ikB_3 + i\epsilon E_1 = \mu h_1,$$

$$C_1 h_0 - i\epsilon h_1 = \mu E_1, \quad -ikh_1 - C_1 h_2 = \mu B_3,$$

$$C_2 h + C_2 B_2 + ikB_1 + i\epsilon E_3 = \mu h_3,$$

$$-C_2h_0 - i\epsilon h_3 = \mu E_3, \quad -C_2h_2 + ikh_3 = \mu B_1,$$
$$-i\epsilon h_0 - ikh_2 + C_3h_1 - C_3h_3 = \mu h,$$
$$-i\epsilon h - ikE_2 + C_3E_1 - C_4E_3 = \mu h_0,$$
$$ikh + i\epsilon E_2 - C_4B_1 - C_3B_3 = \mu h_2,$$
$$-ikh_0 - i\epsilon h_2 = \mu E_2, \quad C_3h_1 + C_4h_3 = \mu B_2,$$

and have the following constraints

$$b_{m-1}f_1(r) = C_3f_3(r), \quad a_mf_3(r) = C_1f_1(r),$$
$$a_{m+1}f_2(r) = C_4f_3(r), \quad b_mf_3(r) = C_2f_2(r). \qquad (4)$$

From Eqs. (4) we derive the second order equations for separate functions:

$$b_{m-1}a_mf_3 = C_1C_3f_3, \quad a_mb_{m-1}f_1 = C_1C_3f_1,$$
$$a_{m+1}b_mf_3 = C_2C_4f_3, \quad b_ma_{m+1}f_2 = C_2C_4f_2.$$

Evidently, within these pairs we can set $C_3 = C_1$, $C_4 = C_2$. Therefore, the differential constraints and second order equations take on the form

$$b_{m-1}f_1(r) = C_1f_3, \quad a_mf_3 = C_1f_1,$$
$$a_{m+1}f_2(r) = C_2f_3, \quad b_mf_3 = C_2f_2;$$
$$[b_{m-1}a_m - C_1^2]f_3 = 0, \quad [a_mb_{m-1} - C_1^2]f_1 = 0,$$
$$[a_{m+1}b_m - C_2^2]f_3 = 0, \quad [b_ma_{m+1} - C_2^2]f_2 = 0. \qquad (5)$$

In explicit form, Eqs. (5) read

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + B - C_1^2\right)f_3 = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - Bm - C_1^2\right)f_1 = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm - B - C_2^2\right)f_3 = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - Bm - C_2^2\right)f_2 = 0.$$

So we get the following identity $C_2^2 = C_1^2 - 2B$ and only three different equations:

$$(1) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + B - C_1^2\right)f_3 = 0,$$

$$(2) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - Bm - C_1^2\right)f_1 = 0,$$

$$(3) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - Bm - C_1^2 + 2B\right)f_2 = 0.$$

Let $B - C_1^2 = X$, then these equations are written in a more symmetrical form

$$(1) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + X\right)f_3 = 0,$$

$$(2) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - B(m+1) + X\right)f_1 = 0,$$

$$(3) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - B(m-1) + X\right)f_2 = 0.$$

With the new variable $x = \frac{Br^2}{2}$, they take on the form

$$(1) \quad \left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{4} - \frac{(m/2)^2}{x^2} + \frac{1}{x}\left(-\frac{m}{2} + \frac{X}{2B}\right)\right]f_3 = 0,$$

$$(2) \quad \left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{4} - \frac{[(m-1)/2]^2}{x^2} + \frac{1}{x}\left(-\frac{m+1}{2} + \frac{X}{2B}\right)\right]f_1 = 0,$$

$$(3) \quad \left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{4} - \frac{[(m+1)/2]^2}{x^2} + \frac{1}{x}\left(-\frac{m-1}{2} + \frac{X}{2B}\right)\right]f_2 = 0.$$

It is sufficient to consider only the first equation in detail. Let us search for solutions in the form $f_3(x) = x^Ae^{Cx}F(x)$, then we readily obtain

$$xF'' + (2A+1+2Cx)F' + \left[\frac{A^2 - (m/2)^2}{x} + 2AC + C - \frac{m}{2} + \frac{X}{2B} + x\left(C^2 - \frac{1}{4}\right)\right]F = 0.$$

Let us impose restrictions

$$A^2 - (m/2)^2 = 0 \Rightarrow A = \pm|m/2|,$$

$$C^2 - \frac{1}{4} = 0 \Rightarrow C + \pm\frac{1}{2}.$$

In order to have the equations referring to bound the states, we should assume

$$A = \pm|m/2|, \quad C = -\frac{1}{2}.$$

This results in the equation of a confluent hypergeometric type

$$xF'' + (|m|+1-x)F' - \left(\frac{|m|+m}{2} + \frac{1}{2} - \frac{X}{2B}\right)F = 0$$

with parameters

$$a = \frac{|m|+m}{2} + \frac{1}{2} - \frac{X}{2B},$$
$$c = |m| + 1, \quad F = \Phi(a, c, x).$$

The polynomial condition $a = -n_1$ leads to

$$(3) \quad \Rightarrow X = +2B\left(\frac{|m|+m}{2} + \frac{1}{2} + n_1\right) > 0,$$
$$n_1 = 0, 1, 2, \ldots.$$

The following solutions correspond to this spectrum

$$(3) \quad f_3(x) = x^{+\frac{|m|}{2}} x^{-x/2} F_1(x),$$
$$F_1(x) = \Phi(-n_1, |m|+1, x).$$

Two other equations give similar results. Thus, we have

$$(3) \quad \begin{aligned} &f_3(x) = x^{+\frac{|m|}{2}} x^{-x/2} F_1(x), \\ &F_3(x) = \Phi(-n_1, |m|+1, x), \\ &X = 2B\left(\frac{|m|+m}{2} + \frac{1}{2} + n_1\right) > B, \\ &n_3 = 0, 1, 2, \ldots \end{aligned} \tag{6}$$

$$(1) \quad \begin{aligned} &f_1(x) = x^{+\frac{|m-1|}{2}} x^{-x/2} F_2(x), \\ &F_1(x) = \Phi(-n_2, |m-1|+1, x), \\ &X = 2B\left(\frac{|m-1|+m+1}{2} + \frac{1}{2} + n_2\right) > B, \\ &n_3 = 0, 1, 2, \ldots \end{aligned} \tag{7}$$

$$(2) \quad \begin{aligned} &f_2(x) = x^{+\frac{|m+1|}{2}} x^{-x/2} F_3(x), \\ &F_2(x) = \Phi(-n_3, |m+1|+1, x), \\ &X = 2B\left(\frac{|m+1|+m-1}{2} + \frac{1}{2} + n_3\right) > B, \\ &n_2 = 0, 1, 2, \ldots \end{aligned} \tag{8}$$

The quantity X in all three cases (6)–(8) should be the same which assumes existence of some correlations within $n-1, n_2, n_3$. Bellow we will apply the simplest quantization rule

$$X = 2BN > 0, \quad N = \left(\frac{|m|+m}{2} + \frac{1}{2} + n\right),$$
$$N = \frac{1}{2}, \frac{3}{2}, \ldots.$$

5. Solving the algebraic system

Let us turn to the algebraic equations

$$\begin{gathered} -i\epsilon h_0 - ikh_2 + C_3 h_1 - C_3 h_3 = -\mu h, \\ -i\epsilon h - ikE_2 + C_3 E_1 - C_4 E_3 = \mu h_0, \\ -C_1 h + C_1 B_2 - ikB_3 + i\epsilon E_1 = \mu h_1, \\ ikh + i\epsilon E_2 - C_4 B_1 - C_3 b_3 = \mu h_2, \\ C_2 h + C_2 B_2 + ikB_1 + i\epsilon E_3 = \mu h_3, \\ C_1 h_0 - i\epsilon h_1 = \mu E_1, \;\; -ikh_0 - i\epsilon h_2 = \mu E_2, \\ -C_2 h_0 - i\epsilon h_3 = \mu E_3, \;\; -C_2 h_2 + ikh_3 = \mu B_1, \\ C_3 h_1 + C_4 h_3 = \mu B_2, \;\; -ikh_1 - C_1 h_2 = \mu B_3. \end{gathered}$$

Recall that

$$C_1 = C_3 = \sqrt{X - B}, \qquad C_2 = C_4 = \sqrt{X + B}.$$

We can present the above system in the matrix form $A\Psi = 0$. As its determinant vanishes we get the equation

$$\begin{gathered} \mu^3(k^2 + \mu^2 - 2X - \epsilon^2)\Big[-2B^2(5k^2 + \mu^2 - 2X - 5\epsilon^2) + \\ +B(-k^2 - \mu^2 + 2X + \epsilon^2)(2\sqrt{X^2 - B^2} - k^2 - \mu^2 + \epsilon^2) - \\ -(k^2 + \mu^2 - 2X - \epsilon^2)^2\Big(\sqrt{X^2 - B^2} - k^2 + \\ +\mu^2 + X + \epsilon^2\Big)\Big] = 0. \end{gathered}$$

This equation is factorized, $P_8 = P_2 P_6$:

$$k^2 + \mu^2 - 2X - \epsilon^2 = 0 \Rightarrow \epsilon^2 - \mu^2 = k^2 - 2X.$$

The second equation with the use of the quantity $W = \epsilon^2 - k^2$ reads as follows

$$\begin{gathered} -W^3 + W^2\Big(-\sqrt{X^2 - B^2} + B + \mu^2 - 5X\Big) + \\ +W\Big[2B\Big(\sqrt{X^2 - B^2} - \mu^2 + X\Big) - \\ -(2X - \mu^2)\Big(2\sqrt{X^2 - B^2} + \mu^2 + 4X\Big) + 10B^2\Big] + \\ +(2X - \mu^2)\Big[B\Big(2\sqrt{X^2 - B^2} - \mu^2\Big) - \\ -(2X - \mu^2)\Big(\sqrt{X^2 - B^2} + \mu^2 + X\Big) + 2B^2\Big] = 0. \end{gathered}$$

With dimensionless variables

$$\frac{W}{\mu^2} \Rightarrow w, \;\; \frac{X}{\mu^2} \Rightarrow x, \;\; \frac{B}{\mu^2} \Rightarrow b, \;\; \frac{\mu}{\mu} \Rightarrow 1,$$

it takes on the form

$$w^3 - w^2\Big(-\sqrt{x^2-b^2}+b+1-5x\Big)-$$

$$-w\Big[2b\Big(\sqrt{x^2-b^2}-1+x\Big)-$$

$$-(2x-1)\Big(2\sqrt{x^2-b^2}+1+4x\Big)+10b^2\Big]+$$

$$+(1-2x)\Big[b\Big(2\sqrt{x^2-b^2}-1\Big)+$$

$$+(1-2x)\Big(\sqrt{x^2-b^2}+1+x\Big)+2b^2\Big]=0.$$

Its analytical solutions are found straightforwardly, but they are helpless. By this reason, let us study its solutions numerically.

Recall that $x = 2bN$. First let us consider the simple case $w = 1-2x = 1-4bN, 1-2x>0$. For several typical examples we find the roots for w (physically interpretable are only positive ones):

$$b = 0.01(0.98, 0.94, 0.9, 0.86, 0.82, 0.78, 0.74, 0.7)^t,$$

$$b = 0.05(0.9, 0.7, 0.5, 0.3, 0.1, -0.1, -0.3, -0.5)^t,$$

$$b = 0.1(0.8, 0.4, 0., -0.4, -0.8, -1.2, -1.6, -2)^t.$$

Now let us examine three roots of the third order equation:

$$b = 0.001\begin{pmatrix} -1 & 0.996002 & 1. \\ -1.00483 & 0.992007 & 0.995996 \\ -1.0089 & 0.988011 & 0.991992 \\ -1.01293 & 0.984015 & 0.987988 \\ -1.01695 & 0.980019 & 0.983984 \\ -1.02096 & 0.976023 & 0.97998 \\ -1.02496 & 0.972027 & 0.975976 \\ -1.02897 & 0.968031 & 0.971972 \end{pmatrix},$$

$$b = 0.01\begin{pmatrix} -1.0002 & 0.960204 & 1. \\ -1.04858 & 0.920701 & 0.959595 \\ -1.0893 & 0.88113 & 0.919181 \\ -1.1296 & 0.841564 & 0.878757 \\ -1.16977 & 0.802008 & 0.838324 \\ -1.20988 & 0.762461 & 0.797879 \\ -1.24996 & 0.722926 & 0.757423 \\ -1.29002 & 0.683403 & 0.716955 \end{pmatrix},$$

$$b = 0.05\begin{pmatrix} -1.00554 & 0.805539 & 1 \\ -1.25012 & 0.619508 & 0.78919 \\ -1.45486 & 0.433263 & 0.576651 \\ -1.65743 & 0.249873 & 0.361149 \\ -1.85931 & 0.0735313 & 0.138565 \\ -2.06088 & -0.0934233-0.0312522i & -0.0934233+0.0312522i \\ -226227 & -0.2929-0.0548385i & -0.2929+0.0548385i \\ -2.46357 & -0.49238-0.0709556i & -0.49238+0.0709556i \end{pmatrix}.$$

In the second case, we can see only two positive roots. In total, we have 3 physically interpretable roots and the corresponding energy series.

Conclusion

For better understanding of the problem, we will consider the nonrelativistic approximation for this model in a separate paper.

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