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Tensor extensions of Lax equations

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Abstract

The Lax equations $dL/dt = [M, L]$ play an important role in the integrability theory of nonlinear evolution equations and quantum dynamics. In this work, tensor extensions of the Lax equations are suggested with $M : V \to V$ and $L : T^k(V) \to V$, $k = 1, 2, \ldots$, on a complex vector space V. These extensions belong to the generalised class of Lax equations (introduced earlier by Bordemann) $dL/dt = \rho_k(M)L$ where ρ_k is a representation of a Lie algebra. The case $k = 1$, $\rho_1 = ad$ corresponds to the usual Lax equations. The extended Lax pairs are studied from the point of view of isomorphic deformations of multilinear structures, conservation laws, exterior algebras and cochain symmetries.

Keywords:

Lax equations, tensor extensions, multilinear algebra, symmetries

Тензорные расширения уравнений Лакса

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Аннотация

Уравнения Лакса $dL/dt = [M, L]$ играют важную роль в теории интегрируемости нелинейных эволюционных уравнений и квантовой динамике. В данной работе предлагаются тензорные расширения уравнений Лакса с $M : V \to V$ и $L : T^k(V) \to V$, $k = 1, 2, \ldots$ на комплексном векторном пространстве V. Эти расширения относятся к обобщенному классу уравнений Лакса (введенному ранее Бордеманном) $dL/dt = \rho_k(M)L$, где ρ_k — представление алгебры Ли. Случай $k = 1$, $\rho_1 = ad$ соответствует обычным уравнениям Лакса. Расширенные пары Лакса изучаются с точки зрения изоморфных деформаций полилинейных структур, законов сохранения, внешних алгебр и коцепных симметрий.

Ключевые слова:

уравнения Лакса, тензорные расширения, полилинейная алгебра, симметрии

Introduction

The idea of symmetry and conservation laws is fundamental in natural sciences. Mathematically, it is reduced to the study of algebraic properties that are invariant under groups of transformations. From this point of view, linear objects are much simpler and more symmetric than nonlinear ones. For instance, nonlinear dynamical systems generally do not admit conservation laws (integrals of the motion) and manifolds of their solutions are much harder to describe than those of linear dynamical systems that always are linear spaces. It is very tempting then to reduce nonlinear dynamical problems to linear problems.

The most remarkable success in this direction is the inverse scattering method of integration of nonlinear evolution equations. The method is based on including the nonlinear evolution into a linear operator L that satisfies a linear evolution equation $dL/dt = [M, L]$ such that the eigenvectors of L satisfy the linear equation with an operator M, while the eigenvalues of L do not evolve. The latter property enables a reconstruction of the nonlinear evolution using a spatial scattering theory for the operator L. For ordinary differential equations, the isospectrality of L is used to find conservation laws of the nonlinear dynamics. The pairs (M, L) are called *Lax pairs*, the equations for the operator L are called *Lax equations* [1-7].

In many cases, useful nonlinear relations exist between solutions to linear dynamical systems. These relations shed extra light to solutions of the relevant nonlinear problems. The simplest nonlinear extension of a linear operator is a multilinear operator. In this work, we realise this idea in replacing the Lax operator L by a multilinear operator that maps solutions to the linear problem with the operator M again to solutions to the same problem. We call the resulting equations *tensor extensions* of the Lax equations.

We show that the extensions thus introduced have a rich algebraic meaning, closely related to the theory of Lie algebras and more general multilinear algebraic structures. We reveal that the extensions we suggest are partial cases of the generalised Lax equations introduced by Bordemann and related to representations of Lie algebras other than the adjoint representation, on which the usual Lax equations are based [8]. Close connections between the solutions to the extended Lax equations and Chevalley-Eilenberg cochain complexes [9, 10] are pointed out. Also, the basic construction presented in this work is another language for description of isomorphic deformations of multilinear algebraic structures on vector spaces with respect to dynamical groups of transformations. In this sense, this work is a continuation of the previous work by the author [11].

1. Basic construction

Let V be a complex vector space and let V_k denote the vector space of k-linear operators $L : V^k \to V$. For any $k = 1,\, 2,\, \ldots$, any linear evolution equation on V

$$dv/dt = Mv, \quad v \in V, \quad M(t) \in V_1 \tag{1}$$

generates the linear evolution equation on V_k

$$dL/dt = \rho_k(M)L,\; L{\in}V_k,\; \rho_k(M(t)) \in End(V_k) \tag{2}$$

such that the solution operator L is a k-symmetry of Eq. (1), i.e., maps k-tuples

$$(v_1(t),\, \ldots,\, v_k(t))$$

of solutions to Eq. (1) again to solutions to Eq. (1). By multiplication with respect to t, we can verify that

$$\rho_k(M)L(v_1, \ldots, v_k) =$$

$$= ML(v_1, \ldots, v_k) - L(Mv_1, v_2, \ldots, v_k) -$$

$$-L(v_1, Mv_2, \ldots, v_k) - \ldots - L(v_1, v_2, \ldots, Mv_k). \tag{3}$$

It is evident that

$$\rho_1(M)L = [M, L], \quad M, L \in V_1,$$

where $[,]$ denotes the commutator, so for $k = 1$ Eq. (2) is the usual Lax equation. Using the canonical injection

$$V^k \to T^k(V)$$

of the Cartesian product V^k into the k-grade of the tensor algebra $T(V)$, due to the universal property of $T(V)$, any solution $L \in V_k$ to Eq. (2) can be uniquely identified with a linear operator $\bar{L} : T^k(V) \to V$. We call the series of Eq. (2), $k = 2,\, 3,\, \ldots$, *tensor extensions* of the Lax equation for $k = 1$.

2. Isomorphic deformations and conservation laws

Eq. (3) enables the solutions to Eqs. (2) for any k and any initial operator $L(0) \in V_k$ to be written in the form

$$L(t)(v_1, \ldots, v_k) = \Phi(t)L(0)(\Phi^{-1}(t)v_1, \ldots, \Phi^{-1}(t)v_k) \tag{4}$$

where $\Phi(t) \in V_1$ is the operator that maps any vector $v \in V$ to the solution $\bar{v}(t) = \Phi(t)v$ to Eq. (1) with the initial value $\bar{v}(0) = v$,

$$d\Phi/dt = M\Phi, \quad \Phi(0) = 1 \in V_1. \tag{5}$$

Due to Eqs. (4), (5), solutions $L(t)$ of the extended Lax equations (2) are k-multiplicative algebraic structures on V that are isomorphic to their initial values $L(0)$ under the evolution of Eq. (1). Eqs. (2) describe then *isomorphic deformations* of k-multiplicative algebraic structures on V. In fact, Eq. (4) is equivalent to

$$L(t)(\Phi(t)v_1, \ldots, \Phi(t)v_k) = \Phi(t)L(0)(v_1, \ldots, v_k). \tag{6}$$

For finite values of time, the fundamental operator $\Phi(t)$ of Eq. (1) is an isomorphism between $L(0)$ and $L(t)$. By the action (6), the group generated by the operators $\Phi(t)$

$$G = \mathrm{gen}\{\Phi(t),\, t \in R\} \subset GL(V) \tag{7}$$

maps any structure $L(0)$ to structures isomorphic to $L(0)$.

Stationary solutions $L(t) = L(0)$ to Eqs. (2) that do not explicitly depend on time describe k-multiplicative structures that are automorphic with respect to the operators $\Phi(t)$ for all values of t,

$$L(0)(\Phi(t)v_1, \ldots, \Phi(t)v_k) = \Phi(t)L(0)(v_1, \ldots, v_k).$$

The group G of Eq. (7) is then a subgroup of the automorphisms group of $L(0)$,

$$G \subset Aut(L(0)).$$

For $k > 1$, evolutions under Eq. (1) on the vector space V generate symmetries of the stationary solutions to Eq. (2) as multiplicative k-linear algebraic structures on V. On the other hand, by definition, solutions to Eq. (2) are k-symmetries of Eq. (1) as they map k-tuples of solutions to Eq. (1) again to solutions to Eq. (1) . We can say that Eqs. (1), (2) describe mutual symmetries of the extended Lax pair (M, L).

The operator $\rho_k(M) : V_k \to V_k$ as a linear function of M defined by Eq. (3) has the property

$$[\rho_k(M), \rho_k(N)] = \rho_k([M, N]) \quad \forall\, M, N \in V_1.$$

Hence, the linear map

$$\rho_k : V_1 \to End(V_k)$$

is a representation of the general Lie algebra $V_1 = \mathfrak{gl}(V)$ on the vector space V_k, i.e., a Lie algebra homomorphism

$$\rho_k : \mathfrak{gl}(V) \to \mathfrak{gl}(V_k). \tag{8}$$

Thus, each Eq. (2) is a partial case of the generalised Lax equation suggested by Bordemann [8]. For the usual Lax equation $k = 1$, we have $\rho_1 = ad$ is the adjoint representation.

By exponentiation, the representation ρ_k of the Lie algebra $\mathfrak{gl}(V)$ generates the linear action (representation) $\bar{\rho}_k$ of the general Lie group $GL(V)$ on the same space V_k. Then any scalar function $f : V_k \to C$ invariant under this action,

$$f(\bar{\rho}_k(m)L) = f(L), \quad \forall\, m \in GL(V),\, L \in V_k, \tag{9}$$

is a conservation law for Eq. (2), i.e., the values $f(L(t))$ are time-independent and do not change along the solutions $L(t)$. It is directly seen by differentiation of Eq. (9) by m at the identity element e of the group $GL(V)$ and the fact that M belongs to the tangent space $T_eGL(V)$. For $k = 1$ and a finite-dimensional vector space V, we have

$$\rho_1 = ad, \quad \bar{\rho}_1(m)L = mLm^{-1}$$

and the trace polynomial functions

$$f_n(L) = \mathrm{Tr}\,(L^n), \quad n = 1, 2, \ldots,$$

are conservation laws for the usual Lax equation.

In fact, functions f satisfying Eq. (9) are conservation laws for Eq. (2) with any operator M. For $k>1$, the explicit description and even existence of such functions is a nontrivial problem even if V is a finite-dimensional vector space. The "isospectrality" of Eq. (2) is closely related to symmetries of the operator M and manifests itself in the following observations.

Let V have a finite dimension N and a basis $v_1,\ldots,v_N$. Any initial operator $L(0)\in V_k$ is defined by its values on the basic vectors of the tensor k-grade $T^k(V)$,

$$L(0)(v_{i_1},\ldots,v_{i_k})=\sum_{s=1}^{N}\lambda^{(s)}_{i_1\ldots i_k}v_s, \tag{10}$$

where the indices $i_1,\ldots,i_k$ independently take all values from the set $\{1,\ldots,N\}$ and $\lambda^{(s)}_{i_1\ldots i_k}$ are complex coefficients, the "structure constants" of the multiplicative algebraic structure $L(0)$. The solution $L(t)$ to Eq. (2) with the initial value $L(0)$ has the property

$$L(t)(\bar v_{i_1}(t),\ldots,\bar v_{i_k}(t))=\sum_{s=1}^{N}\lambda^{(s)}_{i_1\ldots i_k}\bar v_s(t), \tag{11}$$

where $\bar v_j(t)$ are the solutions to Eq. (1) with the initial values v_j and the coefficients $\lambda^{(s)}_{i_1\ldots i_k}$ remain time-independent. This directly follows from Eq. (6) for any k. This does not mean (even for $k=1$) that the structure constants of the initial operator $L(0)$ are conservation laws for the solution $L(t)$. In fact, according to Eq. (4),

$$\begin{aligned}L(t)(v_{i_1},\ldots,v_{i_k})=\sum_{s=1}^{N}\bar\lambda^{(s)}_{i_1\ldots i_k}(t)v_s=\\=\Phi(t)L(0)(\Phi^{-1}(t)v_{i_1},\ldots,\Phi^{-1}(t)v_{i_k}).\end{aligned}$$

The expansion of the initial value by Eq. (10) generates the evolution

$$\lambda^{(s)}_{i_1\ldots i_k}\to\bar\lambda^{(s)}_{i_1\ldots i_k}(t) \tag{12}$$

of the structure constants of $L(0)$ to those of $L(t)$. This evolution is another characteristic of the isomorphism between $L(0)$ and $L(t)$.

The special case where Eq. (2) is explicitly solved is where the operator M is time-independent and the basis $v_1,\ldots,v_N$ is composed of eigenvectors of M with eigenvalues $m_1,\ldots,m_N$. In this case, $\Phi(t)=e^{tM}$ and the group G defined in Eq. (7) is a 1-parameter subgroup of $GL(V)$: $\Phi(t+s)=\Phi(t)\Phi(s)$. According to Eq. (4), the evolution (12) takes the simple form

$$\begin{aligned}\bar\lambda^{(s)}_{i_1\ldots i_k}(t)&=e^{t\phi^{(s)}_{i_1\ldots i_k}}\lambda^{(s)}_{i_1\ldots i_k},\\ \phi^{(s)}_{i_1\ldots i_k}&=m_s-\sum_{p=1}^{k}m_{i_p}.\end{aligned} \tag{13}$$

It follows from Eq. (13) that the structural constants of the initial operator $L(0)$ that satisfy the condition

$$\phi^{(s)}_{i_1\ldots i_k}\lambda^{(s)}_{i_1\ldots i_k}=0$$

do not change under the evolution $L(t)$, i.e., are conservation laws of Eq. (2). In particular, the zero structural constants are always conserved. A nonzero structural constant $\lambda^{(s)}_{i_1\ldots i_k}$ is conserved if the "resonance" $\phi^{(s)}_{i_1\ldots i_k}=0$ takes place between the eigenvalues $m_1,\ldots,m_N$ of the operator M.

The stationary solutions $L(0)$ to Eq. (2) that are automorphic with respect to the group G are defined then by the condition

$$\phi^{(s)}_{i_1\ldots i_k}\lambda^{(s)}_{i_1\ldots i_k}=0\quad\forall s,i_1,\ldots,i_k.$$

It follows, for instance, that if all the eigenvalues are "nonresonant"

$$\phi^{(s)}_{i_1\ldots i_k}\neq 0\quad\forall s,i_1,\ldots,i_k$$

then all stationary solutions to Eq. (2) are trivial $L(0)=0$.

Note that the case $k=2$ with skew-symmetric bilinear operators L corresponds to Lie algebraic structures if additionally the Jacobi identity is satisfied. The finite limit transitions

$$\bar\lambda^{(s)}_{i_1i_2}(t)\to\tilde\lambda^{(s)}_{i_1i_2},\quad t\to\pm\infty,$$

are closely related to Inönü-Wigner contractions and lead to stationary solutions to Eq. (2), automorphic with respect to the "dynamical" group G. This situation has been considered in more detail in the previous work by the author [11].

For $k=1$ (regardless of whether M is time-independent or not), eigenvectors of the operator $L(t)\in V_1$ that evolves under the usual Lax equation are solutions to Eq. (1) and the relevant eigenvalues are time-independent (being eigenvalues of the initial operator $L(0)$). This underlies the inverse scattering method of integration of nonlinear evolution equations [1-7].

3. Exterior algebras and cochain symmetries

It can be verified that, for any $k=1,\,2,\,\ldots$, if $L'\in V_1$ and $L\in V_k$ are solutions to Eq. (2) then the operator composition $L'L\in V_k$ is also a solution to Eq. (2). In this sense, the left multiplication by the solutions to the usual Lax equation is a symmetry of the extended Lax equations (2).

For any M and any k, the operator $\rho_k(M):V_k\to V_k$ is invariant under the action of the symmetric group S_k on V_k. For any permutation $\sigma\in S_k$ of the indices $1,\ldots,k$,

$$\begin{gathered}\rho_k(M)\sigma(L)=\sigma(\rho_k(M)L),\\ \sigma(L)(v_1,\ldots,v_k)\equiv L(v_{\sigma(1)},\,\ldots,\,v_{\sigma(k)}).\end{gathered} \tag{14}$$

Thus, S_k is a symmetry group for Eq. (2). For any solution $L(t)$ and any permutation $\sigma\in S_k$, the "braided" operator $\sigma(L(t))$ is also a solution.

This symmetry and the idea of considering only k-tuples of linearly independent solutions to Eq. (1) leads to the restriction from the infinite-dimensional tensor algebra $T(V)$ to the finite-dimensional exterior (Grassmann) algebra $\bigwedge(V)$ that is a quotient of the tensor algebra with respect to the left-right ideal generated by the tensors of the form $v\otimes v$. In terms of Eq. (2), it means that only alternating k-linear operators L are to be considered, i.e., those with

$$\sigma(L)=\mathrm{sgn}(\sigma)L,\quad\sigma\in S_k.$$

The vector spaces V_k will denote now the vector spaces of alternating operators $L : V^k \to V$. Each such operator can be identified with a linear operator from the k-grade of the exterior algebra, $\bar{L} : \bigwedge^k(V) \to V$. We assume that the vector space V is finite-dimensional, $\dim V = N$.

The construction related to the representation (8) can be extended to a representation of any Lie algebra. In fact, let $\mathfrak{a}$ be a Lie algebra and let

$$\rho : \mathfrak{a} \to \mathfrak{gl}(V) \tag{15}$$

be its representation on V. Then the composition

$$\pi_k = \rho_k \rho : \mathfrak{a} \to \mathfrak{gl}(V_k)$$

is a representation of $\mathfrak{a}$ on V_k. The extended Lax equations (2) are written then as

$$dL/dt = \pi_k(a)L, \quad a \in \mathfrak{a}.$$

Let now the underlying vector space of the Lie algebra $\mathfrak{a}$ be V and the representation ρ in Eq. (15) be the adjoint representation. This enables the Chevalley-Eilenberg cochain complex to be built,

$$V \xrightarrow{\delta} V_1 \xrightarrow{\delta} V_2 \xrightarrow{\delta} \ldots \xrightarrow{\delta} V_N$$

where $\delta : V_{k-1} \to V_k$, $\delta^2 = 0$, is the exterior derivative

$$(\delta L)(v_1, \ldots, v_k) =$$

$$= \sum_{s=1}^{k} (-1)^{s+1} \rho(v_s) L(v_1, \ldots, \hat{v}_s, \ldots, v_k) +$$

$$+ \sum_{s<s'} (-1)^{s+s'} L([v_s, v_{s'}], v_1, \ldots, \hat{v}_s, \ldots, \hat{v}_{s'}, \ldots v_k),$$

$$k > 1, \quad (\delta L)v = \rho(v)L, \quad L \in V.$$

Here $[,]$ is the Lie bracket in $\mathfrak{a}$ and the hat means that the relevant variable should be omitted [9,10]. The solutions $L \in V_k$ to Eq. (2) are then naturally identified with (time-dependent) k-cochains of this complex.

It can be verified that the exterior derivative δ is a symmetry of the set of the extended Lax equations (2). In fact, if $L \in V_k$ is a solution in the k-grade then $\delta L \in V_{k+1}$ is a solution in the next $(k+1)$-grade. We call this symmetry *cochain symmetry*. In the case of the exterior algebra, according to Eqs. (4), (5), for $k = 1, N$ the extended Lax equations (2) are solved explicitly as

$$L(t) = \Phi(t) L(0) \Phi^{-1}(t), \quad k = 1,$$

$$L(t) = \frac{\Phi(t)}{\det \Phi(t)} L(0), \quad k = N.$$

4. Conclusion

It has been demonstrated that the classical Lax equations, important in the integrability theory and quantum dynamics, can be extended in a manner closely related to symmetries of multilinear algebraic structures and representations of Lie algebras other than the adjoint.

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