УДК 539.12
DOI 10.19110/1994-5655-2023-4-77-87

Fermion with three mass parameters in the uniform magnetic field

Фермион с тремя массовыми параметрами во внешнем магнитном поле

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Abstract

Recently, models for a spin $1/2$ particle with two (or three) mass parameters were developed. Specific features of these models are as follows. For corresponding two (or three) bispinors in absence of external fields, separate Dirac-like equations are derived, they differ in masses. However, in presence of external electromagnetic or gravitational fields with non-vanishing Ricci scalar, the wave equation for bispinors does not split into separated equations but makes quite a definite mixing of two (or three) equations arises. In the present paper, the model of a fermion with three mass parameters is studied in presence of the external uniform magnetics field. After performing a diagonalizing transformation, three separate equations are obtained for particles with different anomalous magnetic moments. Their exact solutions and generalized energy spectra are found.

Keywords:

fermion with three mass parameters, magnetic field, anomalous magnetic moment, exact solutions, energy spectrum

Аннотация

Недавно были развиты модели для частицы со спином $1/2$ с двумя (или тремя) массовыми параметрами. Особенности этих моделей заключаются в следующем. Для соответствующих двух (или трех) биспиноров в отсутствие внешних полей выводятся отдельные уравнения дираковского типа, различающиеся массами. Однако при наличии внешних электромагнитных или гравитационных полей с ненулевым скаляром Риччи волновое уравнение для биспиноров не распадается на отдельные уравнения, а возникает связанная система из двух (или трех) уравнений. В настоящей работе исследуется модель фермиона с тремя массовыми параметрами в присутствии внешнего однородного магнитного поля. После диагонализации матрицы смешивания получаются три отдельных уравнения для частиц с разными аномальными магнитными моментами; найдены их точные решения и получены обобщенные энергетические спектры.

Ключевые слова:

фермион с тремя массовыми параметрами, магнитное поле, аномальный магнитный момент, точные решения, энергетический спектр

Introduction

In the context of existence of the similar neutrinos of different masses, we examine a possibility within the theory of relativistic wave equations to describe particles with several mass parameters. In general, existence of more general wave equations than commonly used ones is well known within the Gel'fand-Yaglom formalism – see references [1-5].

In particular, models for a spin 1/2 particle with two and three mass parameters were developed [6–15]. Specific features of these models are as follows. For two (or three) bispinors, in absence of external fields separate Dirac-like equations are derived, they differ in masses. However, in presence of external electromagnetic field or gravitational field with non-vanishing Ricci scalar, the wave equation for bispinors does not split into separated equations, instead a quite definite mixing of two (or three) equations arises. It was shown that generalized equations for Majorana particle with several mass parameters exist as well. Such generalized Majorana equations are not trivial if the Ricci scalar does not vanish.

In the present paper, the model of a fermion with three mass parameters is studied in presence of the external uniform magnetics field. After applying the diagonalizing transformation, three separate equations are obtained effectively for particles with different anomalous magnetic moments. Their exact solutions and generalized energy spectra are found.

1. General theory in presence of electromagnetic and gravitational fields

We start with the system of equations for a fermion with 3 mass parameters [12, 15]:

$$i\gamma^{\alpha}(x)[\partial_{\alpha}+\Gamma_{\alpha}(x)+ieA_{\alpha}(x)]\Phi_1(x)- \\ -M_1\Phi_1(x)+Y_1\Sigma(x)\Phi(x)=0,$$

$$i\gamma^{\alpha}(x)[\partial_{\alpha}+\Gamma_{\alpha}(x)_ieA_{\alpha}(x)]\Phi_2(x)- \\ -M_2\Phi_2(x)+Y_2\Sigma(x)\Phi(x)=0,$$

$$i\gamma^{\alpha}(x)[\partial_{\alpha}+\Gamma_{\alpha}(x)+ieA_{\alpha}(x)]\Phi_3(x)- \\ -M_3\Phi_3(x)+Y_3\Sigma(x)\Phi(x)=0, \quad (1)$$

where the notations are used

$$Y_1=\frac{4c_3}{3M}c_2(\lambda_1-c_2),\ i=1,2,3,\ L=c_1+\frac{c_3}{\sqrt{6}},$$

$$\Phi(x)=L_1\Phi_1(x)+L_2\Phi_2(x)+L_3\Phi_3(x),$$

$$\Sigma(x)=-ieF_{\alpha\beta}\sigma^{\alpha\beta}(x)+\frac{1}{4}R(x),$$

$$L_1=\frac{-L|c_4|^2-L|c_3|^2+c_2^2-c_2(\lambda_2+\lambda_3)+\lambda_2\lambda_3}{Lc_2c_3(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},$$

$$L_2=\frac{-L|c_4|^2-L|c_3|^2+c_2^2-c_2(\lambda_3+\lambda_1)+\lambda_3\lambda_1}{Lc_2c_3(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},$$

$$L_3=\frac{-L|c_4|^2-L|c_3|^2+c_2^2-c_2(\lambda_1+\lambda_2)+\lambda_1\lambda_2}{Lc_2c_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.$$

With the notation $|c_4|=a$, $|c_3|=b$, the mixing matrix in the equation (1) is specified by the relations

$$Y_1L_1=\frac{4}{3M}(\lambda_1-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_2+\lambda_3)+\lambda_2\lambda_3}{L(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},$$

$$Y_1L_2=\frac{4}{3M}(\lambda_1-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_3+\lambda_1)+\lambda_3\lambda_1}{L(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},$$

$$Y_1L_3=\frac{4}{3M}(\lambda_1-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_1+\lambda_2)+\lambda_1\lambda_2}{L(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)},$$

$$Y_2L_1=\frac{4}{3M}(\lambda_2-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_2+\lambda_3)+\lambda_2\lambda_3}{L(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},$$

$$Y_2L_2=\frac{4}{3M}(\lambda_2-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_3+\lambda_1)+\lambda_3\lambda_1}{L(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},$$

$$Y_2L_3=\frac{4}{3M}(\lambda_2-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_1+\lambda_2)+\lambda_1\lambda_2}{L(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)},$$

$$Y_3L_1=\frac{4}{3M}(\lambda_3-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_2+\lambda_3)+\lambda_2\lambda_3}{L(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},$$

$$Y_3L_2=\frac{4}{3M}(\lambda_3-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_3+\lambda_1)+\lambda_3\lambda_1}{L(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},$$

$$Y_3L_3=\frac{4}{3M}(\lambda_3-c_2)\times \\ \times\frac{-L(a^2+b^2)+c_2^2-c_2(\lambda_1+\lambda_2)+\lambda_1\lambda_2}{L(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.$$

We will consider eq. (1) in the cylindrical coordinates and tetrad

$$dS^2=dt^2-dr^2-r^2d\phi^2-dz^2,$$

$$x^{\alpha}=(t,r,\phi,z),$$

$$e^{\beta}_{(a)}(x)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1/r&0\\0&0&0&1\end{pmatrix}.$$

Ricci rotation coefficients are as follows: $\gamma_{ab0}=0,\ \gamma_{ab1}=0,\ \gamma_{122}=-\gamma_{212}=1/r,\ \gamma_{ab3}=0$. The external magnetic field directed along the axis x_3 is determined as follows

$$A_{\phi}=-\frac{1}{2}Br^2,\quad F_{12}(x)=F_{r\phi}=-Br,$$

$$-ieF_{\alpha\beta}(x)=-ieF_{12}(x)\gamma^1(x)\gamma^2(x)=ieB\gamma^1\gamma^2=$$

$$=ieB\begin{pmatrix}-i\sigma_3&0\\0&-i\sigma_3\end{pmatrix}=eB\Sigma.$$

So, the main equation (1) takes the form (we simplify the notation, $eB\Rightarrow B$)

$$\left[i\gamma^0\frac{\partial}{\partial t}+i\gamma^1\frac{\partial}{\partial r}+i\frac{\gamma^2}{r}\left(\frac{\partial}{\partial\phi}+i\frac{Br^2}{2}\right)+i\gamma^3\frac{\partial}{\partial z}\right]\Phi_1- \\ -M\Phi_1+B\Sigma Y_1(L_1\Phi_1+L_2\Phi_2+L_3\Phi_3)=0,$$

$$\left[i\gamma^0\frac{\partial}{\partial t}+i\gamma^1\frac{\partial}{\partial r}+i\frac{\gamma^2}{r}\left(\frac{\partial}{\partial\phi}+i\frac{Br^2}{2}\right)+i\gamma^3\frac{\partial}{\partial z}\right]\Phi_2-$$
$$-M\Phi_2+B\Sigma Y_2(L_1\Phi_1+L_2\Phi_2+L_3\Phi_3)=0,$$
$$\left[i\gamma^0\frac{\partial}{\partial t}+i\gamma^1\frac{\partial}{\partial r}+i\frac{\gamma^2}{r}\left(\frac{\partial}{\partial\phi}+i\frac{Br^2}{2}\right)+i\gamma^3\frac{\partial}{\partial z}\right]\Phi_3-$$
$$-M\Phi_3+B\Sigma Y_3(L_1\Phi_1+L_2\Phi_2+L_3\Phi_3)=0.$$

For three involved bispinors, we will use the following substitutions

$$\Phi_1=e^{-i\epsilon t}e^{im\varphi}e^{ikz}(f_1,f_2,f_3,f_4)^t,$$
$$\Phi_2=e^{-i\epsilon t}e^{im\varphi}e^{ikz}(g_1,g_2,g_3,g_4)^t,$$
$$\Phi_3=e^{-i\epsilon t}e^{im\varphi}e^{ikz}(h_1,h_2,h_3,h_4)^t,$$

where $()^t$ stands for transpose. Then the previous system reads

$$\left[\epsilon\gamma^0+i\gamma^1\frac{\partial}{\partial r}-\frac{\gamma^2}{r}\mu(r)-k\gamma^3-M\right]\Phi_1+$$
$$+(M-M_1)\Phi_1+B\Sigma Y_1(L_1\Phi_1+L_2\Phi_2+L_3\Phi_3)=0,$$
$$\left[\epsilon\gamma^0+i\gamma^1\frac{\partial}{\partial r}-\frac{\gamma^2}{r}\mu(r)-k\gamma^3-M\right]\Phi_2+$$
$$+(M-M_2)\Phi_2+B\Sigma Y_2(L_1\Phi_1+L_2\Phi_2+L_3\Phi_3)=0,$$
$$\left[\epsilon\gamma^0+i\gamma^1\frac{\partial}{\partial r}-\frac{\gamma^2}{r}\mu(r)-k\gamma^3-M\right]\Phi_3+$$
$$+(M-M_3)\Phi_3+B\Sigma Y_3(L_1\Phi_1+L_2\Phi_2+L_3\Phi_3)=0,$$

where

$$\mu(r)=m+\frac{1}{2}Br^2,\quad \Sigma=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}.$$

Using the shortening notations for elements of the mixing matrix

$$Z_{kj}=(BY_k)L_j=d_kL_j,$$

we present the system as follows

$$\widehat{\partial}\Phi_k+Z_{kj}\Phi_j=0,$$
$$\widehat{\partial}=\Sigma^{-1}\left[\epsilon\gamma^0+i\gamma^1\frac{\partial}{\partial r}-\frac{\gamma^2}{r}\mu(r)-k\gamma^3-M\right].$$

The mixing matrix should be reduced to a diagonal form by a linear transformation

$$\overline{\Phi}=S\Phi,\quad \widehat{\partial}\overline{\Phi}=SZS^{-1}\overline{\Phi},$$
$$SZS^{-1}=\begin{pmatrix}\mu_1&0&0\\0&\mu_2&0\\0&0&\mu_3\end{pmatrix}.$$

As a result we get

$$\Sigma^{-1}\widehat{\partial}\begin{pmatrix}\overline{\Phi}_1\\\overline{\Phi}_2\\\overline{\Phi}_3\end{pmatrix}+\begin{pmatrix}\mu_1&0&0\\0&\mu_2&0\\0&0&\mu_3\end{pmatrix}\begin{pmatrix}\overline{\Phi}_1\\\overline{\Phi}_2\\\overline{\Phi}_3\end{pmatrix}=0,$$

or

$$\left[\epsilon\gamma^0+i\gamma^1\frac{\partial}{\partial r}-\frac{\gamma^2}{r}\mu(r)-k\gamma^3-M\right]\overline{\Phi}_1+\mu_1\Sigma\overline{\Phi}_1=0,$$
$$\left[\epsilon\gamma^0+i\gamma^1\frac{\partial}{\partial r}-\frac{\gamma^2}{r}\mu(r)-k\gamma^3-M\right]\overline{\Phi}_2+\mu_2\Sigma\overline{\Phi}_2=0,$$
$$\left[\epsilon\gamma^0+i\gamma^1\frac{\partial}{\partial r}-\frac{\gamma^2}{r}\mu(r)-k\gamma^3-M\right]\overline{\Phi}_3+\mu_3\Sigma\overline{\Phi}_3=0. \quad (2)$$

Thus, after transformation, three separate equations are obtained effectively for particles with different anomalous magnetic moments.

2. Solving the basic equation

Let us briefly describe the procedure for solving the basic equation (2):

$$\left[\epsilon\gamma^0+i\gamma^1\frac{\partial}{\partial r}-\frac{\gamma^2}{r}\sigma(r)-k\gamma^3-M+\Gamma Z\right]\begin{pmatrix}f_1\\f_2\\f_3\\f_4\end{pmatrix}=0.$$

Using the spinor basis for the Dirac matrices [3, 6], we obtain four equations

$$-i\left(\frac{d}{dr}+\mu\right)f_4+(\epsilon+k)f_3+(\Gamma-M)f_1=0,$$
$$-i\left(\frac{d}{dr}-\mu\right)f_3+(\epsilon-k)f_4-(\Gamma+M)f_2=0,$$
$$i\left(\frac{d}{dr}+\mu\right)f_2+(\epsilon-k)f_1+(\Gamma-M)f_3=0,$$
$$i\left(\frac{d}{dr}-\mu\right)f_1+(\epsilon+k)f_2-(\Gamma+M)f_4=0. \quad (3)$$

Let $\dfrac{d}{dr}\pm\mu(r)=D_\pm$. Equations (3) can be considered as two linear subsystems and their solutions are

$$f_1=+i\frac{(\epsilon+k)D_+f_2+(\Gamma-M)D_+f_4}{(\Gamma-M)^2-(\epsilon^2-k^2)},$$
$$f_2=+i\frac{(\epsilon-k)D_-f_1-(\Gamma+M)D_-f_3}{(\Gamma+M)^2-(\epsilon^2-k^2)},$$
$$f_3=-i\frac{(\Gamma-M)D_+f_2+(\epsilon-k)D_+f_4}{(\Gamma-M)^2-(\epsilon^2-k^2)},$$
$$f_4=-i\frac{-(\Gamma+M)D_-f_1+(\epsilon+k)D_-f_3}{(\Gamma+M)^2-(\epsilon^2-k^2)}.$$

Eliminating the variables f_1, f_3 in the equations above, we obtain

$$-\frac{\Gamma+M}{\Gamma-M}f_2+\frac{\epsilon-k}{\Gamma-M}f_4=\frac{D_-D_+f_2}{(\Gamma-M)^2-(\epsilon^2-k^2)}+$$
$$+\frac{\epsilon-k}{\Gamma-M}\times\frac{D_-D_+f_4}{(\Gamma-M)^2-(\epsilon^2-k^2)}, \quad (4)$$

$$f_2 - \frac{\Gamma + M}{\epsilon + k} f_4 = \frac{D_- D_+ f_2}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} + \\ + \frac{\Gamma - M}{\epsilon + k} \times \frac{D_- D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}.$$

Subtract the second equation from the first equation and substitute the resulting expression for f_2 into (4), so we obtain the fourth order equation for f_4:

$$-\frac{d^4 f_4}{dr^4} + \left[\frac{e^2 B^2}{2} r^2 - eB(2m-1) - \right. \\ \left. -2(\Gamma^2 - M^2 - k^2 + \epsilon^2) + \frac{2m(m+1)}{r^2}\right] \frac{d^2 f_4}{dr^2} + \\ + \left[e^2 B^2 r - 4\frac{m(m+1)}{r^3}\right] \frac{df_4}{dr} + \left[-\frac{e^4 B^4}{16} r^4 + \right. \\ + \frac{e^2 B^2}{4}\left[eB(2m-1) + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2)\right] r^2 - \\ -eB(2m-1)(\Gamma^2 - M^2 - k^2 + \epsilon^2) - (\Gamma^2 + M^2 + k^2 - \epsilon^2)^2 + \\ + 4\Gamma^2 M^2 - \frac{e^2 B^2}{4}(6m^2 - 2m - 1) + \\ + \frac{m(m+1)[eB(2m-1) + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2)]}{r^2} - \\ \left. - \frac{m(m-2)(m+3)(m+1)}{r^4}\right] f_4 = 0. \qquad (5)$$

Similarly, we can obtain the fourth order equation for the function f_2. Equations for f_2 and f_4 turn out to be the same. To study the fourth order equation, we will use the factorization method:

$$\widehat{F}_4(r) f(r) = \widehat{f}_2(r) \widehat{g}_2(r) f(r) = 0,$$

$$\widehat{f}_2(r) = \frac{d^2}{dr^2} + P_0 r^2 + P_1 + \frac{P_2}{r^2},$$

$$\widehat{g}_2(r) = \frac{d^2}{dr^2} + Q_0 r^2 + Q_1 + \frac{Q_2}{r^2}.$$

Computing the product

$$\widehat{F}_4 = \left(\frac{d^2}{dr^2} + P_0 r^2 + P_1 + \frac{P_2}{r^2}\right) \times \\ \times \left(\frac{d^2}{dr^2} + Q_0 r^2 + Q_1 + \frac{Q_2}{r^2}\right)$$

and equating the result to the operator (5), we find two solutions for sets of numerical coefficients:

$$I \qquad P_0 = -\frac{1}{4} B^2 e^2, \quad P_2 = -m(m+1),$$

$$P_1 = eB\left(m - \frac{1}{2}\right) + \Gamma^2 - M^2 - \\ - k^2 + \epsilon^2 + 2\Gamma\sqrt{\epsilon^2 - k^2},$$

$$II \qquad Q_0 = -\frac{1}{4} B^2 e^2, \quad Q_2 = -m(m+1),$$

$$Q_1 = eB\left(m - \frac{1}{2}\right) + \Gamma^2 - M^2 - \\ - k^2 + \epsilon^2 - 2\Gamma\sqrt{\epsilon^2 - k^2}.$$

The variant II differs only in sign at the parameter Γ. Thus, we are to solve two second-order differential equations:

$$\left[\frac{d^2}{dr^2} - \frac{B^2 e^2 r^2}{4} + eB\left(m - \frac{1}{2}\right) + \Gamma^2 - M^2 - k^2 + \right. \\ \left. + \epsilon^2 + 2\Gamma\sqrt{\epsilon^2 - k^2} - \frac{m(m+1)}{r^2}\right] f = 0, \qquad (6)$$

$$\left[\frac{d^2}{dr^2} - \frac{B^2 e^2 r^2}{4} + eB\left(m - \frac{1}{2}\right) + \Gamma^2 - M^2 - k^2 + \right. \\ \left. + \epsilon^2 - 2\Gamma\sqrt{\epsilon^2 - k^2} - \frac{m(m+1)}{r^2}\right] g = 0.$$

They differ only in sign at the parameter Γ. Consider the equation (6). Let us make the change of the variable $x = eBr^2/2$. Solutions are constructed in the form $f = x^a e^{bx} F$. Taking into account the constraints $a = -m/2, (m+1)/2$ and $b = -1/2$, we obtain the equation for F:

$$x\frac{d^2 F}{dx^2} + \left(\frac{1}{2} + 2a - x\right)\frac{dF}{dx} -$$

$$-\frac{1}{4eB}\left[eB(4a+1) - 4\Gamma\sqrt{\epsilon^2 - k^2} - (2m-1)eB - \right. \\ \left. -2(\Gamma^2 - M^2 - k^2 + \epsilon^2)\right] F = 0.$$

It is an equation of the confluent hypergeometric type with parameters

$$\gamma = 2a + \frac{1}{2},$$

$$a = -\frac{1}{4eB}\left[eB(4a+1) - 4\Gamma\sqrt{\epsilon^2 - k^2} - (2m-1)eB - \right. \\ \left. -2(\Gamma^2 - M^2 - k^2 + \epsilon^2)\right].$$

To construct solutions corresponding to bound states, one should use the positive values of the parameter a (for definiteness, we assume that $eB > 0$):

$$a = -\frac{m}{2} \;\; (m < 0); \quad a = \frac{m+1}{2} > 0 \;\; (m \geq 0).$$

The polynomial conditions $\alpha = -n$ (let $\epsilon^2 - k^2 = \lambda$) provides us with the quantization rule for energy values

$$a + \frac{1}{2} - \frac{m}{2} + \frac{M^2 - \Gamma^2}{2eB} + n = \frac{\Gamma\sqrt{\lambda}}{eB} + \frac{\lambda}{2eB}.$$

Hence, using the notation $M^2 + 2eB(a + 1/2 - m/2 + n) = N$, we obtain

$$\lambda = (\sqrt{N} - \Gamma)^2 > 0. \qquad (7)$$

From (7) we find the formula for the energy values

$$\epsilon^2 - k^2 = \left[\sqrt{M^2 + 2eB\left(a + \frac{1-m}{2} + n\right)} - \Gamma\right]^2 .$$

Depending on the value of a , we have two expressions for N:

$$m < 0, \quad a = -\frac{m}{2},$$
$$\epsilon^2 - k^2 = \left[\sqrt{M^2 + 2eB(1/2 - m + n)} - \Gamma\right]^2;$$

$$m \geq 0, \quad a = \frac{m+1}{2},$$
$$\epsilon^2 - k^2 = \left[\sqrt{M^2 + 2eB(1 + n)} - \Gamma\right]^2.$$

Thus, we obtain two series of energies

$$I: \quad \lambda = (\sqrt{N} - \Gamma)^2; \quad II: \quad \lambda = (\sqrt{N} + \Gamma)^2.$$

Let us consider a special case of an electrically neutral particle with the magnetic moment (neutron). The transition to the case of a neutral particle can be carried out with the help of simple formal changes, $e \to 0, \ \lambda \to \infty, \ e\lambda \to \Lambda$, so that

$$\Gamma = \lambda \frac{e}{\hbar} \frac{B\hbar}{mc} \Rightarrow \Gamma = \Lambda \frac{B}{mc^2}.$$

No additional calculations are needed. We obtain two second-order equations:

$$\left[\frac{d^2}{dr^2} + (\sqrt{\epsilon^2 - k^2} + \Gamma)^2 - M^2 - \frac{m(m+1)}{r^2}\right] f = 0,$$
$$\left[\frac{d^2}{dr^2} + (\sqrt{\epsilon^2 - k^2} - \Gamma)^2 - M^2 - \frac{m(m+1)}{r^2}\right] f = 0. \quad (8)$$

General solutions of equations (8) have the form

$$f(r) = \sqrt{r}(J_{m+1/2}(x) + Y_{m+1/2}(x)),$$
$$x = \sqrt{\left(\sqrt{\epsilon^2 - k^2} + \Gamma\right)^2 - M^2} r;$$
$$g(r) = \sqrt{r}(J_{m+1/2}(y) + Y_{m+1/2}(y)),$$
$$y = \sqrt{\left(\sqrt{\epsilon^2 - k^2} - \Gamma\right)^2 - M^2} r.$$

From the form of these equations, we can conclude that the magnetic moment manifests itself in an external magnetic field in a quite definite way: in fact, everything reduces to solutions with cylindrical symmetry for an ordinary free particle with spin 1/2, but with a certain replacement

$$\epsilon^2 - M^2 \Rightarrow \left(\sqrt{\epsilon^2 - k^2} \pm \Gamma\right)^2 - M^2.$$

The main manifestation of the magnetic moment of a neutral particle is the modification (spatial scaling) of the wave functions in directions transverse to the magnetic field. Apparently, such a modification of the transverse structure of a neutron beam can be observed experimentally, for example, in neutron Bessel beams. Obviously, the transition to the situation of an electrically neutral particle can also be carried out in the case of a particle with three mass parameters.

3. Diagonalization of the mixing matrix

Let us turn back to the system of three equations

$$\Sigma^{-1}\left[\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \frac{\gamma^2}{r}\mu(r) - k\gamma^3 - M\right]\Phi_1 +$$
$$+(d_1L_1\Phi_1 + d_1L_2\Phi_2 + d_1L_3\Phi_3 + (M - M_1)\Phi_1) = 0,$$
$$\Sigma^{-1}\left[\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \frac{\gamma^2}{r}\mu(r) - k\gamma^3 - M\right]\Phi_2 +$$
$$+(d_2L_1\Phi_1 + d_2L_2\Phi_2 + d_2L_3\Phi_3 + (M - M_2)\Phi_2) = 0,$$
$$\Sigma^{-1}\left[\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \frac{\gamma^2}{r}\mu(r) - k\gamma^3 - M\right]\Phi_3 +$$
$$+(d_3L_1\Phi_1 + d_3L_2\Phi_2 + d_3L_3\Phi_3 + (M - M_3)\Phi_3) = 0,$$

or briefly

$$\widehat{\partial} = \Sigma^{-1}\left[\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \frac{\gamma^2}{r}\mu(r) - k\gamma^3 - M\right],$$
$$\widehat{\partial}\Phi_k + T_{kj}\Phi_j = 0.$$

To find the transformation matrix S, which will diagonalize the mixing matrix, we are to study the following equation $ST = T_0S$, or in detail

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \times$$
$$\times \begin{pmatrix} M - M_1 + d_1L_1 & d_1L_2 & d_1L_3 \\ d_2L_1 & M - M_2 + d_2L_2 & d_2L_3 \\ d_3L_1 & d_3L_2 & M - M_3 + d_3L_3 \end{pmatrix} =$$
$$= \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}.$$

Hence we obtain three linear subsystems:

$$(M - M_1 + d_1L_1)s_{11} + d_2L_1s_{12} + d_3L_1s_{13} = \mu_1 s_{11},$$
$$d_1L_2s_{11} + (M - M_2 + d_2L_2)s_{12} + d_3L_2s_{13} = \mu_1 s_{12},$$
$$d_1L_3s_{11} + d_2L_3s_{12} + (M - M_3 + d_3L_3)s_{13} = \mu_1 s_{13};$$

$$(M - M_1 + d_1L_1)s_{11} + d_2L_1s_{12} + d_3L_1s_{13} = \mu_1 s_{11},$$
$$d_1L_2s_{11} + (M - M_2 + d_2L_2)s_{12} + d_3L_2s_{13} = \mu_1 s_{12},$$
$$d_1L_3s_{11} + d_2L_3s_{12} + (M - M_3 + d_3L_3)s_{13} = \mu_1 s_{13};$$

$$(M - M_1 + d_1L_1)s_{31} + d_2L_1s_{32} + d_3L_1s_{33} = \mu_3 s_{31},$$
$$d_1L_2s_{31} + (M - M_2 + d_2L_2)s_{32} + d_3L_2s_{33} = \mu_3 s_{32},$$
$$d_1L_3s_{31} + d_2L_3s_{32} + (M - M_3 + d_3L_3)s_{33} = \mu_3 s_{33}.$$

Here we have three eigenvalue problems

$$\begin{pmatrix} M-M_1+d_1L_1-\mu_1 & d_2L_1 & d_3L_1 \\ d_1L_2 & M-M_2+d_2L_2-\mu_1 & d_3L_2 \\ d_1L_3 & d_2L_3 & M-M_3+d_3L_3-\mu_1 \end{pmatrix} \begin{pmatrix} s_{11} \\ s_{12} \\ s_{13} \end{pmatrix} = 0,$$

$$\begin{pmatrix} M-M_1+d_1L_1-\mu_2 & d_2L_1 & d_3L_1 \\ d_1L_2 & M-M_2+d_2L_2-\mu_2 & d_3L_2 \\ d_1L_3 & d_2L_3 & M-M_3+d_3L_3-\mu_2 \end{pmatrix} \begin{pmatrix} s_{21} \\ s_{22} \\ s_{23} \end{pmatrix} = 0,$$

$$\begin{pmatrix} M-M_1+d_1L_1-\mu_3 & d_2L_1 & d_3L_1 \\ d_1L_2 & M-M_2+d_2L_2-\mu_3 & d_3L_2 \\ d_1L_3 & d_2L_3 & M-M_3+d_3L_3-\mu_3 \end{pmatrix} \begin{pmatrix} s_{31} \\ s_{32} \\ s_{33} \end{pmatrix} = 0.$$

Note that the three rows of the matrix S may be found only up to arbitrary multipliers. The condition for the existence of solutions for these three systems is the vanishing of the determinant

$$det \begin{pmatrix} M-M_1+d_1L_1-\mu & d_2L_1 & d_3L_1 \\ d_1L_2 & M-M_2+d_2L_2-\mu & d_3L_2 \\ d_1L_3 & d_2L_3 & M-M_3+d_3L_3-\mu \end{pmatrix} = 0.$$

Let us get the explicit form of the third order equation for parameter μ:

$$-\mu^3+(3M-M_1-M_2-M_3+d_1L_1+d_2L_2+d_3L_3)\mu^2+$$
$$+\big[-d_1L_1(2M-M_2-M_3)-d_2L_2(2M-M_1-M_3)-$$
$$-d_3L_3(2M-M_2-M_1)-3M^2+2(M_1+M_2+M_3)M-$$
$$-(M_1M_2+M_1M_3+M_2M_3)\big]\mu+$$
$$+d_1L_1(M-M_2)(M-M_3)+d_2L_2(M-M_3)(M-M_1)+$$
$$+d_3L_3(M-M_1)(M-M_2)+$$
$$+(M-M_1)(M-M_2)(M-M_3)=0.$$

Given the relation $M_i = M/\lambda_i$, the equation can be transformed into the following form

$$-\mu_3+\left[M\left(3-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}-\frac{1}{\lambda_3}\right)+L_1d_1+L_2d_2+\right.$$
$$\left.+L_3d_3\right]\mu^2+\left[-ML_1d_1\left(2-\frac{1}{\lambda_2}-\frac{1}{\lambda_3}\right)-\right.$$
$$-ML_2d_2\left(2-\frac{1}{\lambda_1}-\frac{1}{\lambda_3}\right)-ML_3d_3\left(2-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)-$$
$$-3M^2+2M^2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\right)-$$
$$\left.-M^2\left(\frac{1}{\lambda_1}\frac{1}{\lambda_2}+\frac{1}{\lambda_1}\frac{1}{\lambda_3}+\frac{1}{\lambda_2}\frac{1}{\lambda_3}\right)\right]\mu+$$
$$+\left[M^2L_1d_1\left(1-\frac{1}{\lambda_2}\right)\left(1-\frac{1}{\lambda_3}\right)+\right.$$
$$+M^2L_2d_2\left(1-\frac{1}{\lambda_3}\right)\left(1-\frac{1}{\lambda_1}\right)+$$
$$+M^2L_3d_3\left(1-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\lambda_2}\right)+$$
$$\left.+M^3\left(1-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\lambda_2}\right)\left(1-\frac{1}{\lambda_2}\right)\right]=0. \quad (9)$$

Let us detail the explicit form of the elements of the mixing matrix. First, we take into account that the parametrization of possible values of the mass parameters $M_i = \dfrac{M}{\lambda_i}$ can be simplified. Recall that the roots λ_i are solutions of the characteristic equation

$$\lambda^3-(c_1+c_2)\lambda^2+(c_1c_2+a^2+b^2)\lambda-(c_1a^2+c_2b^2)=0,$$

where $c_3 = b > 0,\ c_4 = a > 0$. Taking into account the identities

$$\lambda_1+\lambda_2 = c_1+c_2-\lambda_3, \quad \lambda_1\lambda_2 = \frac{c_1a^2+c_2b^2}{\lambda_3},$$

we find expressions for λ_1, λ_2 in terms of λ_3 :

$$\lambda_1 = \frac{c_1+c_2-\lambda_3}{2}- -\sqrt{\left(\frac{c_1+c_2-\lambda_3}{2}\right)^2-\frac{c_1a^2+c_2b^2}{\lambda_3}},$$

$$\lambda_2 = \frac{c_1+c_2-\lambda_3}{2}+ +\sqrt{\left(\frac{c_1+c_2-\lambda_3}{2}\right)^2-\frac{c_1a^2+c_2b^2}{\lambda_3}}.$$

The value of the root λ_3 may be arbitrary, because the physically meaningful parameter is $M_3 = M/\lambda_3$ (at an arbitrary M). The simplest expression for λ_3 is obtained when $c_1 = c_2 = 1$. In this case, the cubic equation is simplified

$$\lambda^3-2\lambda^2+(1+k)\lambda-k=0, \quad k=a^2+b^2.$$

Its roots are given by the formulas

$$\lambda_3 = 1, \quad \lambda_1 = \frac{1}{2}-\frac{1}{2}\sqrt{1-4k},$$

$$\lambda_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1-4k}, \quad k \in \left(0, \frac{1}{4}\right).$$

Accordingly, the mass parameters M_i are specified by the formulas

$$M_3 = M, \quad M_1 = \frac{2M}{1-\sqrt{1-4k}},$$

$$M_2 = \frac{2M}{1+\sqrt{1-4k}}, \quad k \in \left(0, \frac{1}{4}\right).$$

Let us construct tables of values for λ_i and $\lambda_i^{-1} = M_i/M,\ i = 1, 2,\ (\lambda_3 = \lambda_3^{-1} = 1)$, depending on the parameter k:

Table 1

Parameter values $\lambda_i = \frac{1}{2}\left(1 \mp \sqrt{1-4k}\right), i = 1, 2$ depending on k

Таблица 1

Значения параметров $\lambda_i = \frac{1}{2}\left(1 \mp \sqrt{1-4k}\right), i = 1, 2$ в зависимости от k

k	0.24	0.22	0.209	0.207	0.205	0.203	0.20	0.16	0.12	0.08	0.04	0.01
λ_1	0.4	0.327	0.298	0.293	0.288	0.283	0.276	0.2	0.139	0.088	0.042	0.01
λ_2	0.6	0.673	0.702	0.707	0.712	0.717	0.724	0.8	0.861	0.912	0.958	0.989

Table 2

Parameter values $\lambda_i^{-1} = 2\left(1 \mp \sqrt{1-4k}\right)^{-1}, i = 1, 2$ depending on k

Таблица 2

Значения параметров $\lambda_i^{-1} = 2\left(1 \mp \sqrt{1-4k}\right)^{-1}, i = 1, 2$ в зависимости от k

k	0.24	0.22	0.209	0.207	0.205	0.203	0.20	0.16	0.12	0.08	0.04	0.01
$1/\lambda_1$	2.5	3.058	3.356	3.413	3.472	3.534	3.623	5	7.194	11.364	23.810	100
$1/\lambda_2$	1.667	1.486	1.425	1.414	1.404	1.395	1.381	1.250	1.161	1.096	1.044	1.011

Let us detail the coefficients L_i in the expression for $\Phi(x)$:

$$\Phi(x) = L_1\Phi_1(x) + L_2\Phi_2(x) + L_3\Phi_3(x),$$

$$L_1 = \frac{-2k}{b}\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \\ + \frac{1}{2Lb}\frac{1-2(\lambda_2+\lambda_3)+4\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},$$

$$L_2 = \frac{-2k}{b}\frac{1}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \\ + \frac{1}{2Lb}\frac{1-2(\lambda_3+\lambda_1)+4\lambda_3\lambda_1}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},$$

$$L_3 = \frac{-2k}{b}\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} + \\ + \frac{1}{2Lb}\frac{1-2(\lambda_1+\lambda_2)+4\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)},$$

or (we take into account that $\lambda_3 = 1$)

$$L_1 = \frac{1}{b}\frac{1}{(\lambda_1-1)(\lambda_1-\lambda_2)}\left\{-2k + \frac{1}{2L}(-1+2\lambda_2)\right\},$$

$$L_2 = \frac{1}{b}\frac{1}{(\lambda_2-1)(\lambda_2-\lambda_1)}\times \\ \times\left\{-2k + \frac{1}{2L}(-1+2\lambda_1)\right\},$$

$$L_3 = \frac{1}{b}\frac{1}{(1-\lambda_1)(1-\lambda_2)}\times \\ \times\left\{-2k + \frac{1}{2L}(1-2\lambda_1-2\lambda_2+4\lambda_1\lambda_2)\right\}, \quad (10)$$

where

$$L = 1 + \frac{b}{\sqrt{6}}, \quad (a^2+b^2) = k < \frac{1}{4}, \quad 0 < 2b < 1.$$

Let us detail the coefficients d_i:

$$d_1 = \frac{4Bb}{6M}\left(\lambda_1 - \frac{1}{2}\right), \quad d_2 = \frac{4Bb}{6M}\left(\lambda_2 - \frac{1}{2}\right),$$

$$d_3 = \frac{4Bb}{6M}\left(\lambda_3 - \frac{1}{2}\right). \quad (11)$$

Combinations d_iL_j appear in the mixing matrix, so the parameter b in the denominators in (10) will be canceled with the parameter b in (11). Also, since the dimension of the quantity B is M^2 (inverse square meter), we can introduce substitutions

$$4B = 6rM^2 \Rightarrow d_i = M\frac{4B}{6M^2}b\left(\lambda_i - \frac{1}{2}\right) = \\ = Mrb\left(\lambda_i - \frac{1}{2}\right) = MD_i,$$

where the quantities D_i are dimensionless. With this in mind, the cubic equation (9) is transformed to the form

$$-\mu^3 + M\left[3 - \frac{1}{\lambda_1} - \frac{1}{\lambda_2} - \frac{1}{\lambda_3} + L_1D_1 + L_2D_2 + \right.$$

$$\left. + L_3D_3\right]\mu^2 + M^2\left[-L_1D_1\left(2 - \frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right) - \right.$$

$$-L_2D_2\left(2 - \frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right) - L_3D_3\left(2 - \frac{1}{\lambda_1} - \right.$$

$$\left. - \frac{1}{\lambda_2}\right) - 3 + 2\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right) -$$

$$-\left(\frac{1}{\lambda_1}\frac{1}{\lambda_2}+\frac{1}{\lambda_1}\frac{1}{\lambda_3}+\frac{1}{\lambda_2}\frac{1}{\lambda_3}\right)\Bigg]\mu+$$

$$+M^3\Bigg[L_1D_1\left(1-\frac{1}{\lambda_2}\right)\left(1-\frac{1}{\lambda_3}\right)+$$

$$+L_2D_2\left(1-\frac{1}{\lambda_3}\right)\left(1-\frac{1}{\lambda_1}\right)+$$

$$+L_3D_3\left(1-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\lambda_2}\right)+$$

$$+\left(1-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\lambda_2}\right)\left(1-\frac{1}{\lambda_3}\right)\Bigg]=0.$$

The roots of this equation can be found in the form $\mu_i = M\Delta_i$, where Δ_i are dimensionless quantities; then the equation for $\Delta_1, \Delta_2, \Delta_3$ takes the form

$$-\Delta^3+\left[3-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}-\frac{1}{\lambda_3}+L_1D_1+L_2D_2+L_3D_3\right]\Delta^2+$$

$$+\Bigg[-L_1D_1\left(2-\frac{1}{\lambda_2}-\frac{1}{\lambda_3}\right)-L_2D_2\left(2-\frac{1}{\lambda_1}-\frac{1}{\lambda_3}\right)-$$

$$-L_3D_3\left(2-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)-3+2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\right)-$$

$$-\left(\frac{1}{\lambda_1}\frac{1}{\lambda_2}+\frac{1}{\lambda_1}\frac{1}{\lambda_3}+\frac{1}{\lambda_2}\frac{1}{\lambda_3}\right)\Bigg]\Delta+$$

$$+\Bigg[L_1D_1\left(1-\frac{1}{\lambda_2}\right)\left(1-\frac{1}{\lambda_3}\right)+$$

$$+L_2D_2\left(1-\frac{1}{\lambda_3}\right)\left(1-\frac{1}{\lambda_1}\right)+$$

$$+L_3D_3\left(1-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\lambda_2}\right)+$$

$$+\left(1-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\lambda_2}\right)\left(1-\frac{1}{\lambda_3}\right)\Bigg]=0.$$

Allow for that $\lambda_3 = 1$, then we get

$$\Delta^3+\left[2-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}+L_1D_1+L_2D_2+L_3D_3\right]\Delta^2+$$

$$+\Bigg[L_1D_1\left(1-\frac{1}{\lambda_2}\right)+L_2D_2\left(1-\frac{1}{\lambda_1}\right)+$$

$$+L_3D_3\left(2-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)+3-2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+1\right)+$$

$$+\left(\frac{1}{\lambda_1}\frac{1}{\lambda_2}+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\Bigg]\Delta-$$

$$-L_3D_3\left(1-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\lambda_2}\right)=0.$$

Since the parameter b enters as a multiplier into D_1, D_2, D_3 and the denominators of the expressions for the coefficients L_1, L_2, L_3, then in combinations

$$L_1D_1+L_2D_2+L_3D_3,\quad L_1D_1,\ L_2D_2,\ L_3D_3$$

this parameter is reduced. With this in mind, we have the identities

$$L_1=\frac{1}{(\lambda_1-1)(\lambda_1-\lambda_2)}\left\{-2k+\frac{1}{2L}(2\lambda_2-1)\right\},$$

$$L_2=\frac{1}{(\lambda_2-1)(\lambda_2-\lambda_1)}\left\{-2k+\frac{1}{2L}(2\lambda_1-1)\right\},$$

$$L_3=\frac{1}{(1-\lambda_1)(1-\lambda_2)}\times$$

$$\times\left\{-2k+\frac{1}{2L}(1-2\lambda_1-2\lambda_2+4\lambda_1\lambda_2)\right\},$$

$$D_1=\frac{r}{2}(2\lambda_1-1),\quad D_2=\frac{r}{2}(2\lambda_2-1),\quad D_3=\frac{r}{2},$$

where

$$\lambda_1=\frac{1}{2}\left(1-\sqrt{1-4k}\right),\quad \lambda_2=\frac{1}{2}\left(1+\sqrt{1-4k}\right),$$

$$k\in\left(1,\frac{1}{4}\right),\quad L=1+\frac{1}{\sqrt{6}},\quad (a^2+b^2)=k<\frac{1}{4}.$$

Also, we should take into account the identities

$$2\lambda_1-1=-\sqrt{1-4k},\quad 2\lambda_2-1=+\sqrt{1-4k},$$

$$\lambda_1-\lambda_2=-\sqrt{1-4k},\quad \lambda_2-\lambda_1=+\sqrt{1-4k},$$

$$\lambda_1+\lambda_2=1,\quad \lambda_1-1=-\lambda_2,\quad \lambda_2-1=-\lambda_1,$$

$$1-2\lambda_1-2\lambda_2+4\lambda_1\lambda_2=4k-1,\quad \lambda_1\lambda_2=k,$$

then we get simpler expressions

$$L_1=\frac{2}{(1+\sqrt{1-4k})\sqrt{1-4k}}\left\{-2k+\frac{1}{2L}\sqrt{1-4k}\right\},$$

$$L_2=\frac{2}{(1+\sqrt{1-4k})\sqrt{1-4k}}\left\{2k+\frac{1}{2L}\sqrt{1-4k}\right\},$$

$$L_3=\frac{1}{k}\left\{-2k+\frac{1}{2L}(4k-1)\right\},$$

$$D_1=-\frac{r}{2}\sqrt{1-4k},\quad D_2=+\frac{r}{2}\sqrt{1-4k},\quad D_3=\frac{r}{2},$$

$$L=1+\frac{b}{\sqrt{6}},\quad 0<b<\frac{1}{2}.$$

In this way, we arrive at the following cubic equation for Δ:

$$\Delta^3+A\Delta^2+B\Delta+C=0,$$

where the coefficients are (take into account the properties of the roots λ_1, λ_2)

$$A=2-\frac{1}{k}+L_1D_1+L_2D_2+L_3D_3,\quad C=L_3D_3,$$

$$B = -L_1 D_1 \frac{1-\sqrt{1-4k}}{1+\sqrt{1-4k}} - \\ - L_2 D_2 \frac{1+\sqrt{1-4k}}{1-\sqrt{1-4k}} + L_3 D_3 \left(2 - \frac{1}{k}\right) + 1.$$

Using identities

$$L_1 D_1 = \frac{2k - l\sqrt{1-4k}}{1+\sqrt{1-4k}} r,$$

$$L_2 D_2 = \frac{2k + l\sqrt{1-4k}}{1-\sqrt{1-4k}} r,$$

$$L_3 D_3 = -\left(1 + l\frac{1-4k}{2k}\right) r,$$

we transform the cubic equation to a simpler form

$$\Delta^3 + \frac{2k-1}{k}\Delta^2 + \left[1 + \left(1 - l\frac{1-4k}{2k}\right) r\right] \Delta + \\ + \left(1 + l\frac{1-4k}{2k}\right) r = 0.$$

Allowing for the expression for the parameter l:

$$l = \frac{1}{2L} = \frac{\sqrt{6}}{2(\sqrt{6}+b)},$$

we obtain

$$\Delta^3 + \frac{2k-1}{k}\Delta^2 + \\ + \left[1 + \left(1 - \frac{\sqrt{6}}{2(\sqrt{6}+b)}\frac{1-4k}{2k}\right) r\right] \Delta + \\ + \left(1 + \frac{\sqrt{6}}{2(\sqrt{6}+b}\frac{1-4k}{2k}\right) r = 0, \quad (12)$$

where

$$0 < k < \frac{1}{4}, \quad 0 < b < 2.$$

For definiteness, we set $b_1 = 0,\ b_2 = 1,\ b_3 = 2$. The parameter r was introduced by the relation $4B = 6rM^2$ from physical considerations, we will assume that the dimensionless parameter r is small. Below we will follow several situations $r = 10^{-5},\ r = 10^{-3},\ r = 1$. The last value $|r| = 1$ corresponds to a very strong magnetic field. Numerical study showed that dependence of the roots Δ_i upon parameter $b \in (0, 2)$ is very insignificant. By this reason below we take the value $b = 0$. Let us construct tables of values for the roots $\Delta_1, \Delta_2, \Delta_3$ equations (12) (see tables 3–5).

Table 3

The values of the roots $\Delta_i, i = 1, 2, 3$ of equation (12) for $b = 0, r = 10^{-5}$

Таблица 3

Значения корней $\Delta_i, i = 1, 2, 3$ уравнения (12) для $b = 0, r = 10^{-5}$

k	0.24	0.22	0.208	0.204	0.20	0.16	0.10	0.06	0.02	0.01
Δ_1	0.667	0.485	0.419	0.400	0.382	0.250	0.127	0.069	0.021	0.010
Δ_2	1.500	2.060	2.389	2.502	2.618	3.999	7.873	14.598	47.979	97.990
Δ_3	-0.00001	-0.00001	-0.00001	-0.00001	-0.00001	-0.00002	-0.00002	-0.00004	-0.00012	-0.00024

Table 4

The values of the roots $\Delta_i, i = 1, 2, 3$ of equation (12) for $b = 0, r = 10^{-3}$

Таблица 4

Значения корней $\Delta_i, i = 1, 2, 3$ уравнения (12) для $b = 0, r = 10^{-3}$

k	0.24	0.22	0.208	0.204	0.20	0.16	0.12	0.08	0.04	0.01
Δ_1	0.670	0.487	0.420	0.401	0.384	0.252	0.164	0.099	0.049	0.021
Δ_2	1.489	2.059	2.388	2.502	2.617	3.999	6.171	10.404	22.957	97.990
Δ_3	-0.001	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.005	-0.012

Finally, with an even greater increase in the parameter r, the roots become complex-valued

Table 5

The values of the roots $\Delta_i, i = 1, 2, 3$ of equation (12) for $b = 0, r = 1$

Таблица 5

Значения корней $\Delta_i, i = 1, 2, 3$ уравнения (12) для $b = 0, r = 1$

k	0.24	0.209	0.206	0.203	0.18	0.12	0.06	0.01
Δ_1	1.265+1.129i	1.588+0.726i	1.625+0.656i	1.662+0.574i	1.173	0.695	0.503	0.406
Δ_2	-0.362	-0.392	-0.395	-0.398	2.804	6.128	14.727	98.221
Δ_3	1.265-1.129i	1.588-0.726i	1.625-0.656i	1.662-0.574i	-0.422	-0.489	-0.653	-0.627

This means that at such magnetic field, the model becomes non-interpretable.

Conclusions

In the present paper, the model of a fermion with three mass parameters is studied in presence of the external uniform magnetic field. After diagonalizing transformation, three separate equations are obtained effectively for particles with different anomalous magnetic moments. Their exact solutions and generalized energy spectra are found. After diagonalizing the mixing matrix, we reduce the problem for three separated Dirac–like equations for particles with different anomalous magnetic moment. It is shown that for a very strong magnetic field, the model becomes non-interpretable, because the effective anomalous moments turns out to be complex-valued.

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For citation:
Ovsiyuk, E.M. Fermion with three mass parameters in the uniform magnetic field / E.M. Ovsiyuk, A.P. Safronov, A.D. Koralkov, Ya.A. Voynova // Proceedings of the Komi Science Centre of the Ural Branch of the Russian Academy of Sciences. Series "Physical and Mathematical Sciences". – 2023. – № 4 (62). – P. 77–87.

Для цитирования:
Овсиюк, Е.М. Фермион с тремя массовыми параметрами во внешнем магнитном поле / Е.М. Овсиюк, А.П. Сафронов, А.Д. Коральков, Я.А. Войнова // Известия Коми научного центра Уральского отделения Российской академии наук. Серия «Физико-математические науки». – 2023. – № 4 (62). – С. 77–87.

Дата поступления статьи: 05.05.2023
Received: 05.05.2023