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Lax equations on Lie superalgebras

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Abstract

It is demonstrated that the standard construction of Lax equations on Lie algebras can be extended to Lie superalgebras, with the even subspace carrying the usual Lax equations. The extended equations inherit the existence of the canonical trace polynomial integrals of motion. An extra set of integrals exists in the odd subspace, with a nontrivial homological structure of the orbit space. This establishes a curious algebraic link between integrable evolution equations, supersymmetry and the deformation theory.

Keywords:

Lie superalgebras, Lax equations, integrals of motion, homological algebra, deformation theory

Уравнения Лакса на супералгебрах Ли

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Аннотация

Показано, что стандартная конструкция уравнений Лакса на алгебрах Ли может быть распространена на супералгебры Ли, в которых четное подпространство несет в себе обычные уравнения Лакса. Расширенные уравнения наследуют существование канонических следовых полиномиальных интегралов движения. В нечетном подпространстве существует дополнительный набор интегралов с нетривиальной гомологической структурой пространства орбит. Это устанавливает любопытную алгебраическую связь между интегрируемыми эволюционными уравнениями, суперсимметрией и теорией деформаций.

Ключевые слова:

супералгебры Ли, уравнения Лакса, интегралы движения, гомологическая алгебра, теория деформаций

Introduction

The substantial interest to graded Lie algebras arose about 60 years ago, in the context of similarity between deformations of complex-analytic structures on compact manifolds and deformations of associative algebras and Lie algebras [1–4], in combination with the relevant cohomological theories [5, 6]. In these algebras, the interplay of "even" and "odd" subspaces carrying skew-symmetric and symmetric multiplication laws plays the crucial role. Later, the new interest to these structures arose in theoretical physics, in the context of supergauge symmetries relating particles of bosonic and fermionic statistics. Although supersymmetry has not been experimentally discovered, these studies stimulated an interesting new mathematics [7–10].

At the same time, the advent of the inverse scattering method gave a boost to the studies of Lie groups and Lie algebras in mathematical physics, in the context of integrability of nonlinear evolution equations. In such studies, the nonlinear dynamics is encoded in the evolution under the "semilinear" Lax equations possessing trace polynomial integrals of motion or revealing the isospectrality of the evolving operators [11–16].

In this work, the standard construction of the Lax equations on Lie algebras is extended to Lie superalgebras, the $\mathbb{Z}_2$-graded Lie algebras of supersymmetry. The extended equations possess the canonical trace polynomial integrals of motion and so can be applied in a similar manner to nonlinear problems. It is shown that the odd subspace admits extra polynomial integrals of motion independent of the canonical integrals. The geometry of the relevant orbit spaces is studied revealing a nontrivial homological algebra. Thus, an algebraic link is established between integrable evolution equations, supersymmetry and the deformation theory. This work can be regarded as a continuation of the previous work by the author [17].

It is assumed that the reader is familiar with the basics of the theory of Lie groups and Lie algebras and their representations as well as the basics of algebraic geometry and homological algebra.

1. Lie superalgebras

The algebra of supersymmetry comes from theoretical physics as an attempt to combine into one unified theory two statistically different types of particles, bosons and fermions. According to the method of second quantization, the (complex finite-dimensional) vector state spaces of these two types are separated by parity, the one being represented in the even space $\mathfrak{l}^0$, the other in the odd space $\mathfrak{l}^1$. To relate these spaces, one assumes that the same symmetry (connected) Lie group G linearly acts on both spaces. The group actions are represented by the group homomorphisms

$$T^k : G \to GL(\mathfrak{l}^k), \quad k = 0, 1.$$

The even action is assumed to be simply the adjoint action of $G, T^0 = Ad$, i.e. $\mathfrak{l}^0$ is the Lie algebra of the Lie group G. The bilinear skew-symmetric bracket in $\mathfrak{l}^0$

$$[,]^0 : \mathfrak{l}^0 \times \mathfrak{l}^0 \to \mathfrak{l}^0, \quad [x, y]^0 = -[y, x]^0$$

is the standard Lie bracket. The differential ad of T^0

$$ad : \mathfrak{l}^0 \to End(\mathfrak{l}^0),\ ad(x)y = [x, y]^0,\ x, y \in \mathfrak{l}^0 \quad (1)$$

represents the adjoint linear action of $\mathfrak{l}^0$ on itself. It is further assumed that the odd action T^1 is tensorially intertwined with T^0. This means that a symmetric bilinear bracket

$$[,]^1 : \mathfrak{l}^1 \times \mathfrak{l}^1 \to \mathfrak{l}^0, \quad [x, y]^1 = [y, x]^1$$

is defined on $\mathfrak{l}^1$ with values in $\mathfrak{l}^0$ such that

$$[T^1(g)x, T^1(g)y]^1 = T^0(g)[x, y]^1,$$
$$x, y \in \mathfrak{l}^1, \quad g \in G. \quad (2)$$

Using the brackets $[,]^k$, $k = 0, 1$, and the differential of the action T^1

$$\rho : \mathfrak{l}^0 \to End(\mathfrak{l}^1), \quad (3)$$

a bilinear bracket $[,]$ on the direct sum

$$\mathfrak{l} = \mathfrak{l}^0 \oplus \mathfrak{l}^1$$

can be defined as

$$[x, y] = \begin{cases} [x, y]^0, & x, y \in \mathfrak{l}^0, \\ [x, y]^1, & x, y \in \mathfrak{l}^1, \\ \rho(x)y, & x \in \mathfrak{l}^0,\ y \in \mathfrak{l}^1. \end{cases}$$

With this bracket, the graded vector space $\mathfrak{l}$ becomes a (complex) *Lie superalgebra*, i.e., a $\mathbb{Z}_2$-graded algebra whose bracket satisfies the conditions

$$\begin{gathered} [x, y] \subseteq \mathfrak{l}^{\xi+\eta}, \quad [x, y] = -(-1)^{\xi\eta}[y, x], \\ (-1)^{\xi\nu}[x, [y, z]] + (-1)^{\xi\eta}[y, [z, x]] + \\ +(-1)^{\eta\nu}[z, [x, y]] = 0 \\ \forall\, x \in \mathfrak{l}^\xi,\ y \in \mathfrak{l}^\eta,\ z \in \mathfrak{l}^\nu,\ \xi, \eta, \nu = 0, 1. \end{gathered} \quad (4)$$

The skew-symmetry between $\mathfrak{l}^0$ and $\mathfrak{l}^1$ and the graded Jacobi identity are externally imposed on $x, y, z \in \mathfrak{l}^1$ (to naturally extend the representation theory) while the rest of the conditions follow the intrinsic properties of the construction above.

The combined action $T = (T^0, T^1)$ of the Lie group G on the Lie superalgebra $\mathfrak{l} = (\mathfrak{l}^0, \mathfrak{l}^1)$ generates the *structural group* of automorphisms of $\mathfrak{l}$,

$$[T(g)x, T(g)y] = T(g)[x, y], \quad x, y \in \mathfrak{l}, \quad g \in G. \quad (5)$$

The differential (ad, ρ) of this action generates a representation of the Lie algebra $\mathfrak{l}^0$ on $\mathfrak{l}$.

2. Representations

Representations of Lie superalgebras are Lie superalgebra homomorphisms

$$\phi : \mathfrak{l} \to L, \quad \phi([x, y]) = [\phi(x), \phi(y)]_L \quad (6)$$

into operator Lie superalgebras L. The latter are constructed as follows. For a $\mathbb{Z}_2$-graded (complex finite-dimensional) vector space

$$V = V^0 \oplus V^1,$$

let L^0, L^1 be the spaces of linear operators $V \to V$ of homogeneous degrees 0,1. This means that operators from L^0 act on the grades while those from L^1 permute the grades,

$$L^0 V^k \subseteq V^k, \quad k = 0, 1, \quad L^1 V^{0,1} \subseteq V^{1,0}.$$

On the $\mathbb{Z}_2$-graded vector space

$$L = L^0 \oplus L^1$$

define a bracket $[,]_L$ by the rule

$$\begin{gathered} [X, Y]_L = XY - (-1)^{\xi\eta} YX, \\ X \in L^\xi,\ Y \in L^\eta,\ \xi, \eta = 0, 1. \end{gathered} \quad (7)$$

With this bracket, L is a Lie superalgebra (the graded Jacobi identity follows from Eq. (7)). Representations of $\mathfrak{l}$ are homomorphisms of Eqs. (6), (7) such that

$$\phi(\mathfrak{l}^k) \subseteq L^k, \quad k = 0, 1. \quad (8)$$

In particular, the restrictions

$$\phi^0 = \phi|_{\mathfrak{l}^0} \quad (9)$$

to the even subspace are representations of the Lie algebra $\mathfrak{l}^0$.

Nontrivial representations of Lie superalgebras always exist. For example, the homomorphism

$$\mathfrak{l} \to Ider(\mathfrak{l}), \quad x \to \partial_x \equiv [x, \cdot] \quad (10)$$

to the space of inner derivations of $\mathfrak{l}$ satisfies the requirement. This representation generalizes the adjoint representation of a Lie algebra. The existence of faithful representations (in a more general context of graded Lie algebras over commutative rings) has been proved in Ref. [3]. Each faithful representation of $\mathfrak{l}^0$ (guaranteed by Ado's theorem) can be extended to a faithful representation of $\mathfrak{l}$.

3. Invariants and Lax equations

The group action T on $\mathfrak{l}$ admits a set of *canonical invariants*, the (complex) trace polynomial functions on $\mathfrak{l}$

$$I_s[\phi](x) = \mathrm{Tr}\left([\phi(x)]^s\right), \quad x \in \mathfrak{l} \quad (11)$$

taken for any power $s \geqslant 0$ and any representation ϕ of $\mathfrak{l}$. In fact, according to Eq. (7), for $X \in L^0$ and $Y \in L$, the bracket $[X, Y]_L$ is the commutator of operators. Hence T acts on operators of algebra representations by conjugation and so preserves the traces of their powers. The restrictions of Eq. (9) generate the set of canonical invariants in the even subspace,

$$I_s^0[\phi^0](x) = I_s[\phi^0](x), \quad x \in \mathfrak{l}^0. \tag{12}$$

These are the standard trace polynomial invariants generated by the Lie bracket in $\mathfrak{l}^0$.

The intertwining of Eq. (2) enables extra invariants to be built for the group action T^1 on $\mathfrak{l}^1$. Precisely, for any invariant f of the action T^0, the function

$$I^1[f](x) = f([x,x]), \quad x \in \mathfrak{l}^1, \tag{13}$$

is an invariant of the action T^1. In fact, the map

$$w : \mathfrak{l}^1 \to \mathfrak{l}^0, \quad w(x) = [x,x] \tag{14}$$

gives a (nonlinear) intertwining of T^1 with T^0. Its composition fw with any invariant f of T^0 is an invariant of T^1. The invariants given by Eq. (13) are called *derived invariants*.

For each $k = 0, 1$, the canonical invariants $I_s[\phi]$ of Eq. (11) are integrals of motion (conservation laws) of evolution equations of the form

$$dl/dt = [m,l], \quad m \in \mathfrak{l}^0, \quad l \in \mathfrak{l}^k \tag{15}$$

where $m = m(t)$ is any time-independent or (continuously) time-dependent magnitude. In fact, for any m, including the case where m depends on l, the trajectories of the solutions to Eq. (15) in the subspaces $\mathfrak{l}^k$ belong to orbits of the group actions T^k determined by the initial values. Eqs. (15) are called *Lax equations on the Lie superalgebra* $\mathfrak{l}$.

In the subspace $\mathfrak{l}^1$, Eq. (15) is rewritten as

$$dl/dt = \rho(m)l, \quad l \in \mathfrak{l}^1 \tag{16}$$

where ρ is the representation of $\mathfrak{l}^0$ on $\mathfrak{l}^1$ given by the differential of the group action T^1 (see Eq. (3)). This is a generalization of the standard Lax equation on the Lie algebra $\mathfrak{l}^0$ to another representation subspace $\mathfrak{l}^1$. Similar generalizations (outside the Lie superalgebras theory) have been considered, for example, in Ref. [16]. The derived invariants $I^1[f]$ given by Eq. (13) are integrals of motion of Eq. (16) additional to the canonical invariants.

The property of the Lax equations to have the "m-universal" conservation laws is very useful. It enables one to integrate nonlinear evolution equations (15) generated by any (continuous) dependences of m on l and t.

4. Geometry of orbits

Since the invariants are integrals of motion, the trajectories of evolution under Eq. (15) belong to the intersections of integral surfaces, on which the invariants take constant values determined by the initial states. Each such intersection is filled with orbits of the group action T on $\mathfrak{l}$. The form of the canonical invariants $I_s[\phi]$ suggests their strong dependence on the representations ϕ. The representations (on the same vector space) are subdivided into equivalence classes with respect to the canonical invariants,

$$\exists g \in G : \ \phi' = \phi T(g) \quad \longrightarrow \quad I_s[\phi'] = I_s[\phi].$$

Besides this, it is hard to formulate anything general about the integral surfaces created by the invariants $I_s[\phi]$.

The derived invariants $I^1[f]$ on the odd subspace $\mathfrak{l}^1$ are different. They are written as compositions of any T^0-invariant with the map w of Eq. (14) that is independent of representations of $\mathfrak{l}$. By Eq. (5), we have

$$[T^1(g)x, T^1(g)x] = T^0(g)[x,x], \qquad \forall\, g \in G, \quad x \in \mathfrak{l}^1. \tag{17}$$

Hence, if the vector $v = [x_0, x_0] \in \mathfrak{l}^0$ is fixed under the group action T^0,

$$T^0(g)v = v \quad \forall g \in G, \tag{18}$$

then the trajectory of the solution $l(t)$ to Eq. (16) starting from x_0 is completely contained in the set

$$S_v = \{x \in \mathfrak{l}^1 : [x,x] = v\}. \tag{19}$$

In fact, in this case, any constant function f is suitable for the derived invariant $I^1[f]$. The space of the vectors v defined by Eq. (18) is the zeroth cohomology group $h^0(G, \mathfrak{l}^0)$ of the group G with coefficients in $\mathfrak{l}^0$. This space also forms the centre of the Lie algebra $\mathfrak{l}^0$.

For any v, the relation that defines the set S_v is quadratically polynomial with respect to the coordinates in $\mathfrak{l}^1$, so the set S_v is an (affine) algebraic variety. By Hilbert's Nullstellensatz, it is defined by the zero locus of a proper ideal in the polynomial ring $\mathbb{C}[\mathfrak{l}^1]$ containing these quadratic polynomials. There is an obvious link of Eq. (19) to the classical problem of intersections of quadrics. The variety S_v is symmetric under the reflection with respect to the origin $x \to -x$. It is non-compact in general: the homotheties $v \to \lambda v$, $x \to \sqrt{\lambda}\, x$ ($\lambda \neq 0$) make the varieties S_v and $S_{\lambda v}$ isoomorphic. In the case $v = 0$, removing the trivial orbit $x = 0$, S_v becomes compact as a projective variety.

The special property of the variety S_v of Eq. (19) is that it lies in the intersection of integral surfaces of *all* canonical polynomial invariants passing through the point x_0. In fact, for any representation ϕ of $\mathfrak{l}$ and any $x \in \mathfrak{l}^1$, in accordance with Eqs. (6), (7), (8),

$$\phi([x,x]) = 2[\phi(x)]^2$$

and any odd power of the operator $\phi(x)$ permutes the even and odd subspaces and so has a zero trace. Hence, we obtain for any integer $s > 0$ and any representation ϕ

$$I_{2s}[\phi](x) = 2^{-s}\mathrm{Tr}\left([\phi([x,x])]^s\right), \qquad I_{2s-1}[\phi](x) = 0, \quad x \in \mathfrak{l}^1. \tag{20}$$

According to Eqs. (17), (18), the set S_v is filled with orbits of the group action T^1 on $\mathfrak{l}^1$. This generates the orbit space S_v/G that classifies points of S_v. Two points belong to the same equivalence class if they belong to the same orbit. In the case $v \neq 0$, we will assume that the group action T^1 is irreducible on $\mathfrak{l}^1$.

The classification problem S_v/G can be approached as follows. For any $x \in S_v$, let ∂_x be the inner derivation defined by the homomorphism of Eq. (10). In other words,

$$\partial_x y = [x,y], \quad x \in S_v, \quad y \in \mathfrak{l}.$$

It follows from the Jacobi identity (see Eq. (4)) that

$$2\partial_x\partial_x y = [v, y]. \qquad (21)$$

Eq. (18) is equivalent to the condition

$$[m, v] = 0 \quad \forall m \in \mathfrak{l}^0.$$

By Eq. (21), this implies

$$\partial_x\partial_x \mathfrak{l}^0 = 0. \qquad (22)$$

Also, we have

$$[v, y] = [[y, y], y] = 0 \quad \forall y \in S_v$$

where we again used the Jacobi identity. Hence, S_v is a subset of the centralizer of v in $\mathfrak{l}^1$. This centralizer is G-invariant because v is G-fixed. For $v \neq 0$, we assumed that the T^1-action is irreducible, so the whole $\mathfrak{l}^1$ centralizes v (otherwise, there would exist a smaller invariant subspace of T^1),

$$[\mathfrak{l}^1, v] = 0. \qquad (23)$$

By Eqs. (21), (23), we conclude then that

$$\partial_x\partial_x \mathfrak{l}^1 = 0. \qquad (24)$$

Combination of Eqs. (22), (24) gives

$$\partial_x\partial_x \mathfrak{l} = 0.$$

Considering the restrictions on the even and odd subspaces

$$\partial_x^k = \partial_x|_{\mathfrak{l}^k}, \quad k = 0, 1,$$

we have on $\mathfrak{l}$

$$\partial_x^1\partial_x^0 = \partial_x^0\partial_x^1 = 0.$$

This enables the Lie superalgebra $\mathfrak{l}$ to be represented as the "loop" chain complex

$$\mathfrak{l}^0 \underset{\partial_x^1}{\overset{\partial_x^0}{\rightleftarrows}} \mathfrak{l}^1$$

with respect to the differential ∂_x. Introducing the kernels and images (the cycles and boundaries)

$$Z_x^k = ker\, \partial_x^k, \quad B_x^{0,1} = im\, \partial_x^{1,0},$$

we assign to each point $x \in S_v$ the even and odd homology groups as the quotients

$$H_x^k = Z_x^k / B_x^k, \quad k = 0, 1. \qquad (25)$$

The groups H_x^k, $H_{x'}^k$ are isomorphic if x, x' belong to the same G-orbit.

Introducing the vector spaces

$$Z_x = Z_x^0 \oplus Z_x^1,\ B_x = B_x^0 \oplus B_x^1,\ H_x = H_x^0 \oplus H_x^1,$$

we see that Z_x is a Lie superalgebra that is a subalgebra of $\mathfrak{l}$, B_x is an ideal in Z_x and so $H_x = Z_x/B_x$ also becomes a Lie superalgebra.

The subspaces Z_x^1, B_x^1 are respectively the tangent space to S_v and the tangent space to the orbit of x at the point x. If the odd homology group is trivial, $H_x^1 = 0$, then the orbit of x covers a whole neighbourhood of the point x in S_v. All small deformations of x within S_v are G-orbit equivalent. Such points x are called *rigid*. For $Z_x^1 = 0$ (for $v \neq 0$), the set S_v consists of one point x (which in this case is fixed under the group action, $B_x^1 = 0$). If $H_x^1 \neq 0$ then the orbit of x tends to lie strictly inside S_v.

The subspace Z_x^0 is the Lie subalgebra of $\mathfrak{l}^0$ that centralizes x: $[Z_x^0, x] = 0$. The subspace B_x^0 is the image of x under the odd inner derivations: $B_x^0 = [\mathfrak{l}^1, x]$. By the Jacobi identity and Eqs. (19), (23), it is a Lie subalgebra (actually an ideal) of Z_x^0. If the even homology group is trivial, $H_x^0 = 0$, then x is a *simple point* of S_v. In fact, let $H_x^0 = 0$ and let $x + u \in S_v$ be a deformation of the point x in S_v. Then u satisfies the *deformation equation*

$$2\partial_x^1 u + [u, u] = 0. \qquad (26)$$

We can write the solution to Eq. (26) as a formal power series

$$u = zu_1 + z^2u_2 + \dots \qquad (27)$$

in some (complex) scalar parameter z. The first term

$$2\partial_x^1 u_1 = 0 \quad \longrightarrow \quad u_1 \in Z_x^1$$

can be chosen arbitrarily. To find the higher terms, the following induction can be applied. Let the first q terms be known. Then they satisfy the equations

$$\begin{gathered} 2\partial_x^1 u_r + J_r = 0, \\ J_r = \sum_{p=1}^{r-1} [u_p, u_{r-p}], \quad r = 1, \dots, q. \end{gathered} \qquad (28)$$

To find the $(q+1)$th term, the following equation should be solved

$$2\partial_x^1 u_{q+1} + J_{q+1} = 0. \qquad (29)$$

Let

$$u^{(q)} = \sum_{r=1}^{q} z^r u_r$$

be the qth partial sum. Using the Jacobi identity, we have

$$[x + u^{(q)}, [x + u^{(q)}, x + u^{(q)}]] = 0.$$

Taking the $(q+1)$th power of z, with the use of Eqs. (23), (24), we obtain

$$[x, J_{q+1}] + \sum_{r=1}^{q} [u_{q+1-r}, 2[x, u_r] + J_r] = 0.$$

By Eqs. (28), this gives

$$\partial_x^0 J_{q+1} = 0.$$

This means that $J_{q+1} \in Z_x^0$ and so $J_{q+1} \in B_x^0$ because we assumed $H_x^0 = 0$. Then Eq. (29) can be resolved for u_{q+1}, uniquely if we take the zero projection to Z_x^1. Hence, all the terms of the power series of Eq. (27) can be uniquely found. This series converges for any u_1 as long as $|z|$ is sufficiently small. We obtain that the point $x \in S_v$ can be analytically deformed within S_v in any direction given by the space Z_x^1 of tangent vectors to S_v. Thus, for $H_x^0 = 0$, the point x is simple. A structure of a complex manifold on S_v can be defined in a neighbourhood of x. The situation is very similar to that described in Ref. [1].

The consideration of dimensions gives the following relations

$$dim\, H_x^k = dim\, \mathfrak{l}^k - \sum_{p=0,1} dim\, B_x^p,$$
$$dim\, Z_x^k = dim\, \mathfrak{l}^k - dim\, B_x^{(k+1)},\ k=0,1. \tag{30}$$

Here in addition to Eq. (25) we used the isomorphisms

$$B_x^{0,1} \simeq \mathfrak{l}^{1,0}/Z_x^{1,0}.$$

Also, since $v \in B_x^0 \subseteq Z_x^0$, we obtain

$$v \neq 0 \quad \longrightarrow \quad dim\, Z_x^0 \geqslant dim\, B_x^0 \geqslant 1 \tag{31}$$

(in particular, if $S_v \neq \emptyset$ for $v \neq 0$ then the group action T^1 on $\mathfrak{l}^1$ cannot be free). It immediately follows from Eq. (30) that if the subspaces $\mathfrak{l}^{0,1}$ are not isomorphic, $dim\, \mathfrak{l}^1 \neq dim\, \mathfrak{l}^0$ (i.e., the representation spaces of ad and ρ are not isomorphic as vector spaces), then the groups $H_x^{0,1}$ are not simultaneously trivial and are not isomorphic, $dim\, H_x^1 \neq dim\, H_x^0$. This is valid for each point $x \in S_v$. This means, for instance, that neither point $x \in S_v$ can be simultaneously a simple point of the variety S_v and have its orbit covering the whole neighbourhood of x in M_v. In particular, for $dim\, \mathfrak{l}^1 \neq dim\, \mathfrak{l}^0$, the variety S_v cannot be a (nontrivial smooth) homogeneous space of the G-action.

Eqs. (30), (31) enable an estimation of possible orbit classes in the space S_v/G to be made. The existence of functions on $\mathfrak{l}^1$ that separate orbits in S_v and their links to the homology on S_v are interesting open questions. In addition to Eq. (20), note that, for the "adjoint representation" of Eq. (10), the canonical integrals take the zero values on S_v,

$$\phi(x) = \partial_x \quad \longrightarrow \quad I_s[\phi](x) = 0,\ x \in S_v,\ s > 0.$$

In fact, for $x \in S_v$, we have $\partial_x^2 = 0$, so the operator ∂_x is nilpotent and its all positive powers have a zero trace.

5. Conclusion

We have shown that the well-known construction of the Lax equations on Lie algebras can be extended to Lie superalgebras, important in mathematics and theoretical physics in their relation to the deformation theory and supersymmetry. Like the usual Lax equations, the extended ones admit the canonical trace polynomial integrals of motion which can be used in the integrability theory for nonlinear evolution equations. Besides the canonical integrals, the extra set of derived integrals occurs in the odd subspace, as a result of tensorial intertwining with the even subspace. This new feature is due to the symmetric character of multiplication within the odd subspace. The orbit spaces generated by constant values of the derived integrals $[x,x] = v$, where v belongs to the 0th cohomology group of the underlying Lie group action, possess the natural (co)homological structure with respect to the inner derivations ∂_x. This structure is generically nontrivial, giving obstacles for the integral surfaces to be locally homogeneous spaces. These results algebraically relate the integrability theory of evolution equations with supersymmetry and the deformation theory.

The future work can be focused on possible connections of the orbit space S_v/G with the "intrinsic properties" of the algebraic variety S_v independent of its embedding into the odd subspace $\mathfrak{l}^1$ (say, in the spirit of the Zariski and Mumford theories). An extension of the described algebraic structures to the general graded Lie algebras should be possible in terms of their natural grading into the even and odd subspaces. From the point of view of physical applications, it can be interesting to relate the above constructions to integrable nonlinear dynamics and supersymmetry (for instance, to connect Eqs. (19), (26) to symplectic geometry and Hamiltonian dynamics as well as to the extended supersymmetry theory, say, for the Poincaré algebra). Possible relations of the deformation Eq. (26) to the Maurer-Cartan formalism and the gauge theories can also be interesting.

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