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# Homological invariants in gauge theories

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# Гомологические инварианты в калибровочных теориях

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## Abstract

**Extending the gauge formalism of the physical field theory to general graded Lie algebras, we show that in this formalism cohomology groups naturally arise, invariant under gauge transformations. Links of these groups to the Chern-Weil theory of characteristic classes are established. Applications of these cohomologies to Gerstenhaber-Nijenhuis deformations and Yang-Mills equations are discussed. These results can also be useful in the theory of integrable evolution equations and geometry of Lie groups.**

## Keywords:

**gauge theories, algebraic formalism, homological invariants**

## Аннотация

**Распространяя калибровочный формализм физической теории поля на общие градуированные алгебры Ли, мы показываем, что в этом формализме естественным образом возникают группы когомологий, инвариантные относительно калибровочных преобразований. Устанавливаются связи этих групп с теорией характеристических классов Чжэня–Вейля. Обсуждаются приложения этих когомологий к деформациям Герстенхабера-Нийенхейса и уравнениям Янга-Миллса. Эти результаты могут быть полезны также в теории интегрируемых эволюционных уравнений и геометрии групп Ли.**

## Ключевые слова:

**калибровочные теории, алгебраический формализм, гомологические инварианты**

## Introduction

The gauge formalism of the physical field theory is an important tool that regulates redundant degrees of freedom and utilizes symmetries of the Lagrangian. It is considered as a basis for a unified theory of physical interactions.

The gauge formalism has two basic mathematical components, geometric and algebraic. Geometrically, gauge fields are associated to connections on principal fibre bundles over the space-time, with the structure group being the symmetry group. Algebraically, the gauge formalism is based on the theory of graded Lie algebras of exterior differential forms on manifolds, with values in a Lie algebra [1].

In these notes, the abstract algebraic component of the gauge theories, leaving aside their geometric features, is applied to general graded Lie algebras. The basic notations of the gauge formalism find their general algebraic analogues. It is shown that to each abstract gauge field a cohomology group can be naturally associated, which is an invariant of the gauge group. We also show that isomorphism classes of these groups are closely related to the Chern-Weil theory of characteristic classes. These homological invariants are applied to a curvature-preserving deformation theory, in the sense of the Gerstenhaber-Nijenhius formalism [2, 3], and to the study of solutions to the Yang-Mills equations [1].

The results obtained can be useful also in the theory of integrable evolution equations (existence of integrable hierarchies [4] on non-Euclidean manifolds) and geometry of Lie groups (Maurer-Cartan forms [5]).

## 1. Gauge theories on graded Lie algebras

Let $\Omega$ be a graded Lie algebra over $\mathbb{R}$, i.e., a $\mathbb{Z}$-graded real vector space

$$\Omega = \bigoplus_{k\in\mathbb{Z}} \Omega^k$$

with a bilinear operation (bracket)

$$[,] : \Omega \times \Omega \to \Omega$$

that is graded skew-symmetric, respects the grading and satisfies the graded Jacobi identity

$$\begin{gathered} [\xi, \eta] = -(-1)^{kl}[\eta, \xi], \quad [\xi, \eta] \in \Omega^{k+l}, \\ (-1)^{kp}[\xi, [\eta, \theta]] + (-1)^{lk}[\eta, [\theta, \xi]] + \\ +(-1)^{pl}[\theta, [\xi, \eta]] = 0, \\ \xi \in \Omega^k, \quad \eta \in \Omega^l, \quad \theta \in \Omega^p. \end{gathered} \tag{1}$$

By Eq. (1), graded Lie algebras are not Lie algebras in the usual sense (although the 0th grade $\Omega^0$ and the even part $\bigoplus \Omega^{2k}$ are usual Lie algebras). The terminology we use is induced by the gauge formalism and the deformation theory (see, for instance, Refs. [2, 5]). In the context of supersymmetry (basically in the even-odd $\mathbb{Z}/2$ setting), algebras with brackets, satisfying conditions (1), are called graded Lie superalgebras (see, for instance, Ref. [3]). In the context of usual Lie algebras, graded Lie algebras are understood as usual Lie algebras, carrying grading.

Elements of the grade $\Omega^k$ are called homogeneous elements of degree $k$. Let $E^k$ be the space of homogeneous operators on $\Omega$ of degree $k$, i.e., endomorphisms of $\Omega$ which shift the grades by $k$,

$$E^k = \{A \in End(\Omega) : A\Omega^l \subseteq \Omega^{l+k},\ l \in \mathbb{Z}\}.$$

Then the endomorphisms space is written as a graded vector space

$$End(\Omega) \equiv E = \bigoplus_{k\in\mathbb{Z}} E^k.$$

Define a bracket on $E$ by the rule

$$[A, B]_0 = AB - (-1)^{ab} BA,$$
$$A \in E^a, \quad B \in E^b.$$

With this bracket the space $E$ of endomorphisms of $\Omega$ becomes a graded Lie algebra. In fact, the bracket $[,]_0$ satisfies relations similar to Eq. (1).

Let $\rho$ denote the adjoint representation of the algebra $\Omega$,

$$\rho : \Omega \to E, \quad \rho(\xi)\eta = [\xi, \eta],$$
$$\rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)]_0. \tag{2}$$

We assume that $\rho$ is faithful,

$$\ker \rho = 0, \tag{3}$$

i.e., the algebra $\Omega$ has trivial centre.

Suppose a (real) Lie group $G$ acts on the algebra $\Omega$ by automorphisms, i.e., there is a representation

$$T : G \to GL(\Omega),$$
$$\rho(T(g)\xi) = S(g)\rho(\xi). \tag{4}$$
$$g \in G, \quad \xi \in \Omega.$$

Here $S$ is the action of $G$ on the algebra $E$ by conjugation automorphisms,

$$S(g) : A \mapsto T(g)AT(g^{-1}),$$
$$g \in G, \quad A \in E.$$

Further, the elements of $\Omega^1$ are called *gauge fields*.

Let $d$ be a differential on $\Omega$, i.e., a derivation of degree 1 that squares to zero,

$$\rho(d\xi) = [d, \rho(\xi)]_0, \quad \xi \in \Omega,$$
$$d \in E^1, \quad dd = 0. \tag{5}$$

To each gauge field $\omega \in \Omega^1$ we associate the linear operator

$$d_\omega = d + \rho(\omega) \in E^1, \tag{6}$$

which we call the *covariant derivative* along $\omega$, and the magnitude

$$\phi(\omega) = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2, \tag{7}$$

which we call the *curvature* (or the *gauge field strength*) of $\omega$. As a consequence of Eqs. (2), (5), for each gauge field, the covariant derivative is a derivation of degree 1,

$$\rho(d_\omega \xi) = [d_\omega, \rho(\xi)]_0, \quad \xi \in \Omega,$$

which is connected to the curvature by the relations

$$d_\omega(\phi(\omega)) = 0, \quad d_\omega d_\omega = \rho(\phi(\omega)). \tag{8}$$

*Proposition 1.* Suppose there exists a smooth function

$$\nu : G \to \Omega^1$$

such that

$$\rho(\nu(g)) = S(g)d - d. \tag{9}$$

Then, for all $g \in G$, the map

$$K(g) : \Omega^1 \to \Omega^1,$$
$$\omega \mapsto T(g)\omega + \nu(g) \tag{10}$$

acts on covariant derivatives by the rule

$$d_{K(g)\omega} = S(g)d_\omega. \tag{11}$$

*Proof.* This is a simple consequence of Eqs. (60), (9). □

Note that, since the adjoint representation $\rho$ is faithful, the function $\nu(g)$ (if exists) is unique. The map (10) is called the *gauge transformation* corresponding to $g \in G$. Two gauge fields $\omega$, $\omega'$ are called *equivalent* (or *gauge equivalent*), $\omega \sim \omega'$, if they are connected by a gauge transformation, $\omega' = K(g)\omega$, for some $g \in G$.

*Corollary 1.* It follows from Eqs. (60), (9) that the function $\nu$ satisfies the property

$$\nu(hg) = K(h)\nu(g).$$

Hence, the gauge transformations form a Lie group,

$$K(hg) = K(h)K(g), \quad h, g \in G,$$

which we call the *gauge group* and denote $K(G)$.

*Corollary 2.* Acting on both sides of Eq. (11) by the operator $d_{K(g)\omega}$ and using Eq. (8), we see that gauge transformations act on curvatures by the rule

$$\phi(K(g)\omega) = T(g)\phi(\omega).$$

*Corollary 3.* By Eq. (9),

$$d_{\nu(g)} = S(g)d,$$

where, for each $g \in G$, the right-hand side squares to zero. Hence, for each $g \in G$, the covariant derivative $d_{\nu(g)}$ is a differential on $\Omega$. Since the adjoint representation $\rho$ is faithful, by Eq. (8), the curvature of $\nu(g)$ is zero. This gives

$$d_{\nu(g)}d_{\nu(g)} = 0, \quad \phi(\nu(g)) = 0.$$

By Eq. (10), $\nu(g)$ is equivalent to the zero gauge field,

$$\nu(g) \sim 0, \quad g \in G.$$

The gauge fields $\nu(g)$ are called *pure gauge fields*.

The assumption of Proposition 1, Eq. (9), is fulfilled at least in two cases: where the differential $d$ is an inner derivation or where $T(g)$ are inner automorphisms. In the former case, we have $d = \rho(\eta)$ for some $\eta \in \Omega^1$ (with $[\eta, \eta] = 0$), so $\nu(g) = T(g)\eta - \eta$. In the latter case, the differential $t_e$ of the representation $T$ at the identity element $e$ of the group $G$ maps the Lie algebra $\mathfrak{g}$ of the group $G$ to the Lie algebra

$\rho(\Omega^0)$ of inner derivations of degree 0, i.e., defines a representation

$$t_e : \mathfrak{g} \to \rho(\Omega^0) \subseteq E^0$$

(accompanied with a homomorphism $\mathfrak{g} \to \Omega^0$, as $\rho$ is faithful). According to Eq. (9), we have $\nu(e) = 0$. The infinitesimal form near the identity element of the group $G$ of the right-hand side of Eq. (9) is

$$[t_e(a), d]_0, \quad a \in \mathfrak{g}.$$

In the case where $t_e(a)$ is an inner derivation for all $a \in \mathfrak{g}$, using the fact that inner derivations form an ideal $\mathrm{ider}(\Omega)$ in the algebra of all derivations $\mathrm{der}(\Omega) \subseteq E$, we obtain that the right-hand side of Eq. (9) is infinitesimally realised as an inner derivation. Since thie right-hand side is a derivation for all $g \in G$, this local analysis is extended to the whole group $G$ in the standard way, by left translations within $G$. We obtain then that, if $T(g)$ are inner automorphisms, the right-hand side of Eq. (9) forms an inner derivation for all $g \in G$. Such a situation realises, for instance, in the special case of the natural action of the Lie algebra $\Omega^0$ on $\Omega$. The latter can be extended to an inner action of the (local) Lie group that has $\Omega^0$ as its Lie algebra.

Note that one of derivations $d$ or $t_e(a)$ is not necessarily inner, as the ideal $\mathrm{ider}(\Omega) \subseteq \mathrm{der}(\Omega)$ is not necessarily prime. Generally speaking, Eq. (9) means that the $S$-action preserves the coset generated by the derivation $d$ in the quotient $\mathrm{der}(\Omega)/\mathrm{ider}(\Omega)$. The case where $T(g)$ are inner automorphisms takes place, for instance, in the field-theoretical gauge formalism.

Eq. (3) suggests that we introduced an abstract nonabelian gauge theory. The abelian case corresponds to the trivial bracket $[,] = 0$, which gives $\rho = 0$. This reduces to the situation where the representation $S$ preserves the differential, $S(g)d = d$ (for instance, $T$ is trivial), the covariant derivative coincides with $d$ for all gauge fields, $d_\omega = d$, the curvature reduces to $\phi(\omega) = d\omega$, and the function $\nu(g)$ is chosen to have zero curvature, $d(\nu(g)) = 0$ for all $g \in G$, and to satisfy the group property of Corollary 1.

## 2. Homological invariants

The differential $d$ makes $\Omega$ a cochain complex and generates the graded cohomology group

$$H = \ker d / \mathrm{im}\, d.$$

The differential $d_{\nu(g)}$ also makes $\Omega$ a cochain complex with the graded cohomology group

$$H(\nu(g)) = \ker d_{\nu(g)} / \mathrm{im}\, d_{\nu(g)}.$$

By Eq. (11), the $T(g)$-action is a quasi-isomorphism of these cochain complexes, so the cohomologies $H(\nu(g))$ and $H$ are isomorphic,

$$H(\nu(g)) \sim H, \quad g \in G.$$

This observation is generalised as follows.

*Proposition 2.* For each gauge field $\omega \in \Omega^1$, the covariant derivative $d_\omega$ is a differential on the graded Lie subalgebra

$$\Omega(\omega) = \ker\,(d_\omega d_\omega) \subseteq \Omega \tag{12}$$

and generates on $\Omega(\omega)$ the graded cohomology group

$$H(\omega) = \ker d_\omega / \mathrm{im}\, d_\omega.$$

Equivalent gauge fields have isomorphic cohomologies,

$$\Omega(\omega) \sim \Omega(\omega'), \quad H(\omega) \sim H(\omega'),$$
$$\omega' = K(g)\omega, \quad g \in G.$$

*Proof.* Since $d_\omega$ commutes with $d_\omega d_\omega$, it maps the kernel $\Omega(\omega)$ to itself and acts on $\Omega(\omega)$ as a differential. The $T(g)$-action realises a quasi-isomorphism of the cochain complexes for gauge-equivalent $\omega$, $\omega'$. □

Note that the subalgebra $\Omega(\omega)$ is never zero. It is indeed a graded Lie subalgebra of $\Omega$, as it coincides with the centralizer of the curvature $\phi(\omega)$. In fact, by Eqs. (1), (8), the curvature $\phi(\omega)$ itself belongs to $\Omega(\omega)$. By definition, more generally, we have $\ker d_\omega \subseteq \Omega(\omega)$. In the special case $\phi(\omega) = 0$, the subalgebra (12) coincides with the ambient algebra,

$$\Omega(\omega) = \Omega, \quad \phi(\omega) = 0.$$

As mentioned, the pure gauge fields $\nu(g)$ all have zero curvature. They are all equivalent to $\omega = 0$, and the cohomology groups $H(\nu(g))$ are all isomorphic to the cohomology group generated by the differential $d$. In general, however, $H(\omega) \not\sim H$, even for $\phi(\omega) = 0$.

By Proposition 2, the cohomology groups $H(\omega)$ are invariants of the gauge group $K(G)$. The cohomologies $H(\omega)$ classify points of the orbit space $\Omega^1/K(G)$. For $H(\omega) \not\sim H(\omega')$, the gauge fields $\omega$ and $\omega'$ are not equivalent, and their $K(G)$-orbits are different.

For each gauge field $\omega$, the cohomology group $H(\omega)$ forms a graded Lie algebra. In fact, for each differential $D$, the image $\mathrm{im}\, D$ is an ideal in the kernel $\ker D$. This follows from the fact that $D$ is a derivation of $\Omega$, i.e., $\rho(D\xi) = [D, \rho(\xi)]_0$, for all $\xi \in \Omega$, and $DD = 0$. Then it is easy to verify that the cohomology class of the bracket $[\xi, \eta]$, where $\xi$, $\eta \in \ker D$, depends only on the cohomology classes of $\xi$, $\eta$. Hence, the bracket $[,]$ in the algebra $\Omega(\omega)$ generates a bracket in the cohomology group $H(\omega)$. This bracket inherits the properties (1), i.e., it is again graded skew-symmetric, respects the grading and satisfies the graded Jacobi identity.

In this context, the quadratic map

$$f : \ker d_\omega \cap \Omega^1(\omega) \to \ker d_\omega \cap \Omega^2(\omega),$$
$$f : u \mapsto [u, u]$$

generates a quadratic map of cohomology groups

$$f' : H^1(\omega) \to H^2(\omega).$$

By Eq. (8), $\phi(\omega) \in \ker d_\omega \cap \Omega^2(\omega)$, so the cohomology class

$$[\phi(\omega)] \in H^2(\omega)$$

can be associated to each gauge field $\omega \in \Omega^1$. By Proposition 1 and Corollary 2, the property of $\omega$ to have trivial or a nontrivial cohomology class $[\phi(\omega)]$ is gauge-invariant. In fact, the condition $\phi(\omega) = d_\omega(\xi(\omega))$ implies

$$\phi(K(g)\omega) = T(g)(\phi(\omega)) =$$
$$= T(g)\{d_\omega(\xi(\omega))\} = d_{K(g)\omega}\{T(g)(\xi(\omega))\}.$$

These observations are useful for the applications below.

## 3. Chern classes

Isomorphism classes of the even cohomology groups $H^{2k}(\omega)$ can be put into the context of Chern characteristic classes as follows.

Let a graded real vector space $\bar\Omega = \bigoplus \bar\Omega^k$ be given that generates a cochain complex with a differential $\bar d$ and the relevant cohomology group $\bar H = \bigoplus \bar H^k$. For each gauge field $\omega$, consider the cochain complex $\Omega(\omega)$ with the differential $d_\omega$ introduced in Section 2. Let

$$p_\omega : \Omega(\omega) \to \bar\Omega$$

be a "quasi-cochain map", i.e., a linear map that respects the grading and maps $d_\omega$-closed elements to to $\bar d$-closed elements,

$$q_\omega d_\omega = \bar d p_\omega \tag{13}$$

with some map $q_\omega : \Omega(\omega) \to \bar\Omega$. Considering the curvature $\phi(\omega) \in \Omega^2(\omega)$, it follows from Eqs. (8), (13) that the magnitude $p_\omega(\phi(\omega)) \in \bar\Omega^2$ is $\bar d$-closed. Hence, it generates the cohomology class

$$c_1(\omega) = [p_\omega(\phi(\omega))] \in \bar H^2,$$

which we call the *first Chern class* of $\omega$ in $\bar H$.

The terminology is induced by the classical Chern-Weil theory of characteristic classes, which links differential geometry and algebraic topology and plays an important part in topology of principal fiber bundles and vector bundles. In the space of connections $\omega$ on such a bundle, with the adjoint gauge group action, $\bar\Omega$ is the de Rham complex of the base manifold, and $p_\omega$ is generated by the Chern-Weil homomorphism (see, for example, [6] and references therein).

The following result shows that the first Chern class is an invariant of gauge transformations and in certain cases depends only on the class $[\phi(\omega)] \in H^2(\omega)$ introduced in Section 2. In fact, $c_1(\omega)$ is a characteristic class of isomorphisms of the cohomology groups $H(\omega)$.

*Proposition 3.* The first Chern class is gauge invariant,

$$c_1(\omega') = c_1(\omega), \quad \omega' = K(g)\omega, \quad g \in G,$$

and under one of the conditions

i) $p_\omega$ is a cochain map, i.e., in Eq. (13) $q_\omega = p_\omega$, or

ii) $\omega$ is a solution to the Yang-Mills equation, $d^*_\omega(\phi(\omega)) = 0$ (see Section 5),

$$c_1(\omega) = [p_\omega[\phi(\omega)]], \quad [\phi(\omega)] \in H^2(\omega). \tag{14}$$

*Proof.* Under the gauge transformations, the relevant cochain complexes, the quasi-cochain maps, the covariant derivatives and the curvatures are transformed as

$$\Omega(\omega') = T(g)\Omega(\omega), \quad p_{\omega'}T(g) = p_\omega, \quad q_{\omega'}T(g) = q_\omega,$$
$$d_{\omega'}T(g) = T(g)d_\omega, \quad \phi(\omega') = T(g)\phi(\omega).$$

This implies

$$p_{\omega'}(\phi(\omega')) = p_\omega(\phi(\omega)),$$

and we obtain that $c_1(\omega)$ is gauge invariant. Further, under condition i), the cochain map $p_\omega$ maps cohomology classes to cohomology classes. This implies Eq. (14). Under condition ii), we have $\phi(\omega) \perp \operatorname{im} d_\omega$ (see Section 5), so $\phi(\omega)$ has zero projection to the space $d_\omega\Omega^1(\omega)$. This implies $\phi(\omega) = [\phi(\omega)]$ and leads again to Eq. (14). □

Note that if the maps $p_\omega$ are $T$-invariant,

$$p_\omega T(g) = p_\omega,$$

then $p_\omega \equiv p$ can be chosen independently of $\omega$.

*Corollary 4.* Under one of conditions i) or ii) of Proposition 3,

$$[\phi(\omega)] = 0 \in H^2(\omega) \quad \longrightarrow \quad c_1(\omega) = 0 \in \bar H^2.$$

In particular, gauge fields of zero curvature have trivial first Chern class. For example, pure gauge fields satisfy this condition.

Higher Chern classes can be introduced as follows. Let the bracket $[,]$ be generated by some associative bilinear operation $\wedge$, i.e.,

$$[\xi,\eta] = \xi\wedge\eta - (-1)^{xy}\eta\wedge\xi, \quad \xi\in\Omega^x,\ \eta\in\Omega^y.$$

We assume that the operation $\wedge$ preserves the automorphisms action $T(G)$, the differential $d$ remaining a derivation of degree 1 that squares to zero,

$$(T(g)\xi)\wedge(T(g)\eta) = T(g)(\xi\wedge\eta),,$$
$$d(\xi\wedge\eta) = (d\xi)\wedge\eta + (-1)^x\xi\wedge d\eta, \quad dd = 0.$$

Using the notation

$$\phi(\omega)^k = \phi(\omega)\wedge\ldots\wedge\phi(\omega) \ \ (k \text{ times}),$$

for each $k = 1,\ 2,\ \ldots$, we define the *$k$th Chern class* to be

$$c_k(\omega) = [p_\omega(\phi(\omega)^k)] \in \bar H^{2k}.$$

Eq. (8) is generalized as

$$d_\omega(\phi(\omega)^k) = 0,$$

so $p_\omega(\phi(\omega)^k)$ are $\bar d$-closed, and the Chern classes are well-defined.

Like the first Chern class $c_1$, the higher Chern classes are also gauge-invariant, being characteristic classes of isomorphisms of the cohomologies $H(\omega)$. Under one of conditions i) or ii) of Proposition 3, Eq. (14) generalizes to

$$c_k(\omega) = [p_\omega[\phi(\omega)^k]], \quad [\phi(\omega)^k] \in H^{2k}(\omega).$$

Classical Chern classes $c'_k(\omega)$ are defined in a different manner, being homogeneous polynomial combinations of the classes $c_k(\omega)$ we introduced. This is related to formal expansions [6]

$$\det(1 + t\phi(\omega)) = 1 + tc'_1(\omega) + t^2c'_2(\omega) + \ldots$$

Here the first Chern class (up to scaling related to integer cohomologies) coincides with that we introduced,

$$c'_1(\omega) = c_1(\omega).$$

## 4. Gerstenhaber-Nijenhuis deformations

The cohomology groups $H^1(\omega)$, $H^2(\omega)$ play an important role in the curvature-preserving deformations, in the general framework of the Gerstenhaber-Nijenhuis theory [2, 3], as described below.

Let $M(\bar\phi)$ be the manifold of gauge fields of a fixed curvature $\bar\phi$,

$$M(\bar\phi) = \{\omega \in \Omega^1 : \phi(\omega) = \bar\phi\}.$$

By Eqs. (8), (12), we have

$$\omega - \omega' \in \Omega^1(\omega) = \Omega^1(\omega'), \quad \omega,\, \omega' \in M(\bar\phi).$$

*Proposition 4.* Let $\omega \in M(\bar\phi)$ have trivial second cohomology group, $H^2(\omega) = \{0\}$, and $\ker d_\omega \cap \Omega^1(\omega) \neq \{0\}$. Then $\omega$ can be deformed within the manifold $M(\bar\phi)$ by a formal power series

$$\begin{gathered} \omega' = \omega + \lambda u_1 + \lambda^2 u_2 + \ldots \in M(\bar\phi), \\ \lambda \in \mathbb{R}, \quad u_k \in \Omega^1(\omega), \end{gathered} \tag{15}$$

in the direction of any tangent vector $u_1 \in \ker d_\omega$.

*Proof.* The power series (15) is supposed to solve the equation

$$d\omega' + \frac{1}{2}[\omega', \omega'] = \bar\phi,$$

which at $\lambda = 0$ is solved by the chosen $\omega \in M(\bar\phi)$. We have formally

$$\begin{gathered} \omega' = \omega + u, \quad d_\omega u + \frac{1}{2}[u, u] = 0, \\ u = \lambda u_1 + \lambda^2 u_2 + \ldots \end{gathered}$$

The first coefficient $u_1$ satisfies the equation

$$d_\omega u_1 = 0$$

and can be chosen arbitrarily from $\ker d_\omega$. Then the higher order coefficients $u_2$, $u_3$, etc., are found recurrently as follows. Let the first $q$ coefficients be known. Then we have

$$\begin{gathered} d_\omega u_r + \frac{1}{2} J_r = 0, \quad r = 1, \ldots, q, \\ J_r = \sum_{p=1}^{r-1} [u_p, u_{r-p}]. \end{gathered} \tag{16}$$

The next $(q+1)$th coefficient should satisfy the equation

$$d_\omega u_{q+1} + \frac{1}{2} J_{q+1} = 0, \tag{17}$$

and we should show that Eq. (2) is solvable. Let

$$u^{(q)} = \sum_{r=1}^{q} \lambda^r u_r$$

denote the $q$th partial sum of the series (15). We obtain from Eq. (1)

$$[\omega + u^{(q)}, [\omega + u^{(q)}, \omega + u^{(q)}]] = 0.$$

Taking the $(q+1)$th power in $\lambda$ of the expression above, this gives

$$[\omega, J_{q+1}] + \sum_{r=1}^{q} [u_{q+1-r}, 2[\omega, u_r] + J_r] = 0.$$

By Eq. (16),

$$2[\omega, u_r] + J_r = -2 du_r,$$

which gives

$$[\omega, J_{q+1}] - 2 \sum_{r=1}^{q} [u_{q+1-r}, du_r] = 0. \tag{18}$$

We have

$$\begin{aligned} dJ_{q+1} = \sum_{r=1}^{q} \left( [du_r, u_{q+1-r}] - [u_r, du_{q+1-r}] \right) = \\ = -2 \sum_{r=1}^{q} [u_{q+1-r}, du_r], \end{aligned}$$

so Eq. (18) becomes

$$[\omega, J_{q+1}] + dJ_{q+1} = d_\omega J_{q+1} = 0.$$

Hence, we obtain $J_{q+1} \in \ker d_\omega \cap \Omega^2(\omega)$. By assumption, $H^2(\omega) = \{0\}$, which implies $J_{q+1} \in \operatorname{im} d_\omega \cap \Omega^2(\omega)$, i.e., Eq. (2) is indeed has a solution. The coefficient $u_{q+1}$ can be chosen uniquely if we require that it has zero projection to $\ker d_\omega$. $\square$

By Proposition 4, the cohomology group $H^2(\omega)$ obstructs the existence of smooth deformations of $\omega$ within the fixed-curvature manifold $M(\phi(\omega) = \bar\phi)$. Indeed, for $H^2(\omega) \neq \{0\}$, Eq. (2) may be unsolvable and not all coefficients $u_q$ of the power series (15) may exist. The subspace $\ker d_\omega \cap \Omega^1(\omega)$ consists of vectors tangent to the manifold $M(\bar\phi)$ at the point $\omega$. If this subspace is trivial then $M(\bar\phi)$ locally consists of one point $\omega$. The condition $\ker d_\omega \cap \Omega^1(\omega) \neq \{0\}$ in Proposition 4 is certainly satisfied if the first cohomology group is nontrivial, $H^1(\omega) \neq \{0\}$.

## 5. Yang-Mills equations

Another application of the cohomology group $H^2(\omega)$ is found in the Yang-Mills theory [1], as described below.

Let a $T$-invariant inner product $\langle , \rangle$ exist on $\Omega$,

$$\langle T(g)\xi, T(g)\eta \rangle = \langle \xi, \eta \rangle, \quad g \in G.$$

This is true, for instance, if $\Omega$ is a pre-Hilbert space and the group $G$ is compact. In this case, the inner product $\langle , \rangle'$ on $\Omega$ is averaged over the group $G$ to a $T$-invariant inner product,

$$\langle \xi, \eta \rangle = \int_G \langle T(g)\xi, T(g)\eta \rangle' \, \partial g.$$

Here $\partial g$ is a left-invariant measure on $G$.

Further we assume that homogeneous elements of different degrees are orthogonal,

$$\langle \Omega^k, \Omega^{k'} \rangle = 0, \quad k \neq k'.$$

By Corollary 2, the inner product $\langle , \rangle$ generates the gauge-invariant energy (or action) functional

$$\mathcal{E}(\omega) = \|\phi(\omega)\|^2 = \langle \phi(\omega), \phi(\omega) \rangle,$$
$$\mathcal{E}(K(g)\omega) = \mathcal{E}(\omega), \quad \omega \in \Omega^1, \quad g \in G.$$

Gauge fields $\omega$ of zero curvatures $\phi(\omega) = 0$ provide the global minimum $\mathcal{E}(\omega) = 0$ of the energy functional.

Consider the local minimisation problem

$$\omega : \mathcal{E}(\omega) \to \min \tag{19}$$

in terms of the Euler-Lagrange formalism. The solutions to the problem (19) are gauge fields $\omega$, such that their small deformations infintesimally preserve the value of the energy functional. By definitions (6), (7), we have, for all $u \in \Omega^1$,

$$\mathcal{E}(\omega + u) - \mathcal{E}(\omega) = 2\langle d_\omega u, \phi(\omega)\rangle + O(\|u\|^2) =$$
$$= 2\langle u, d_\omega^*(\phi(\omega))\rangle + O(\|u\|^2).$$

Here $d_\omega^*$ is the operator adjoint to $d_\omega$,

$$\langle d_\omega \xi, \eta \rangle = \langle \xi, d_\omega^* \eta \rangle, \quad \xi, \eta \in \Omega.$$

Solutions to the problem (19) satisfy the Euler-Lagrange equation

$$d_\omega^*(\phi(\omega)) = 0. \tag{20}$$

The nonlinear Eq. (20) is called the *Yang-Mills equation*.

The existence of the adjoint operator $d_\omega^*$ is guaranteed, for instance, if $\Omega$ is a Hilbert space and $d_\omega$ is bounded. In a more general context, Eq. (20) can be replaced by the condition $\phi(\omega) \perp \operatorname{im} d_\omega$.

*Proposition 5*. Let $\omega$ be a solution to Eq. (20) with trivial cohomology class $[\phi(\omega)] = 0 \in H^2(\omega)$ (and so trivial first Chern class $c_1(\omega) = 0$). Then $\omega$ is of zero curvature, $\phi(\omega) = 0$, and so provides the global minimum of the energy functional (and then all Chern classes are trivial, $c_k(\omega) = 0$).

*Proof*. By assumption, we have $d_\omega^*(\phi(\omega)) = 0$ and $\phi(\omega) = d_\omega(\xi(\omega))$ for some $\xi(\omega) \in \Omega^1(\omega)$. Then

$$\mathcal{E}(\omega) = \langle d_\omega(\xi(\omega)), \phi(\omega)\rangle = \langle \xi(\omega), d_\omega^*(\phi(\omega))\rangle = 0.$$

This implies $\phi(\omega) = 0$. □

The statement inverse to Proposition 5 is also true. The solutions to Eq. (20) with zero curvature $\phi(\omega) = 0$ obviously have trivial cohomology class $[\phi(\omega)] = 0$.

*Corollary 5*. It follows from Proposition 5 that solutions $\omega$ to the Yang-Mills equation with a nonzero curvature $\phi(\omega) \neq 0$ must have nontrivial cohomology classes $[\phi(\omega)] \neq 0$, and hence the second cohomology must be nontrivial, $H^2(\omega) \neq \{0\}$. If $H^2(\omega) = \{0\}$, and $\phi(\omega) \neq 0$, then $\omega$ cannot be a solution to the Yang-Mills equation. Local minima $\omega$ of the energy functional, $\mathcal{E}(\omega) \neq 0$, have nontrivial cohomology classes $[\phi(\omega)] \neq 0$.

Note that, in the field-theoretical gauge formalism, local minima of the energy functional are important because the existence of global minima $\phi(\omega) = 0$ can be topologically obstructed.

Due to the $T$-invariance of the inner product $\langle , \rangle$, if $\omega$ is a solution to Eq. (20) then $K(g)\omega$ is also a solution (with the same energy), for all $g \in G$. Thus, the gauge group $K(G)$ acts on the space $\Omega^1_{\mathrm{YM}}$ of solutions to the Yang-Mills equation. It follows from the results of Section 3 that the cohomology groups $H(\omega)$ classify points of the orbit space $\Omega^1_{\mathrm{YM}}/K(G)$.

Along with the first equation of Eq. (8), the Yang-Mills Eq. (20) defines harmonic curvatures with respect to the Laplacian $\Delta_\omega = d_\omega d_\omega^* + d_\omega^* d_\omega$,

$$\Delta_\omega(\phi(\omega)) = 0 \quad \text{iff}$$
$$d_\omega(\phi(\omega)) = 0, \quad d_\omega^*(\phi(\omega)) = 0.$$

This reveals an analogy with the Hodge theory.

## 6. Conclusion

We have shown that a local part of the gauge formalism of the physical field theory can be formulated purely algebraically, for any graded Lie algebra. Here gauge fields, gauge groups, covariant derivatives and curvatures/field strengths find their general algebraic analogues. In this framework, cohomology groups naturally arise, which are gauge-invariant and encode a useful structural information. Isomorphism classes of these groups can be described in the spirit of the Chern-Weil theory of characteristic classes.

Two applications have been discussed: curvature-preserving deformations, closely related to the Gerstenhaber-Nijenhuis formalism [2, 3], and solutions to the Yang-Mills equations [1]. In the first case, nontriviality of the second cohomology group is an obstruction to existence of smooth deformations, while in the second one, this nontriviality is necessary for existence of local minima of the energy functional. The results presented have maximal generality, valid for any differential and any gauge-invariant inner product on the algebra.

Besides gauge theories, zero-curvature manifolds are encountered also in the theory of nonlinear evolution equations integrable by the inverse scattering transform. The first application (Proposition 4) reveals (co)homological obstructions to existence of integrable hierarchies (such as, for instance, the Ablowitz-Kaup-Newell-Segur hierarchy [4]) on non-Euclidean manifolds. Another example is the Maurer-Cartan forms in geometry of Lie groups [5]. Here Proposition 4 can be useful in the relevant deformation theory.

Note finally that the special case of zero differential $d = 0$ was considered before in the context of cohomologies and deformations of associative algebras and Lie algebras [2]. This has been recently applied also to a study of homological structure of orbit spaces for Lax equations on Lie superalgebras [3].

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