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Vector particle with polarizability in the uniform magnetic field

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Abstract

There is a 15-component equation, which describes the vector particle with the additional electromagnetic characteristics – polarizability. We specify this equation in cylindrical coordinates and in presence of the external uniform magnetic field. After separating the variables, the system of 15 first-order differential equations in the polar coordinate is derived. To resolve this system, we apply the algebraization method. In this approach, the complete wave function is decomposed into the sum of three parts. Dependence of the components in each part is determined by only one corresponding function $F_i(r), i = 1, 2, 3$. We construct these three basic variables in terms of the confluent hypergeometric functions. There is a quantization rule for some spectral parameter exists. Additionally, there arises an algebraic homogenous system of 15 equations, which completely determines the structure of 15-component solutions. From vanishing the determinant of this linear system, we derive a cubic algebraic equation with respect to the energy parameter ε^2. Its solutions are found in analytical form and studied numerically. In this way, we have obtained three energy spectra. One does not depend on the polarizability parameter and the other two are substantially modified by this characteristics.

Keywords:

spin 1 particle, polarizability, cylindrical symmetry, external uniform magnetic field, separation of the variables, algebraization method, exact solutions, energy spectra

Векторная частица с поляризуемостью в однородном магнитном поле

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Аннотация

Известно 15-компонентное уравнение, описывающее векторную частицу с дополнительной электромагнитной характеристикой – поляризуемостью. Это уравнение исследуется в цилиндрических координатах при наличии внешнего однородного магнитного поля. После разделения переменных получена система 15 дифференциальных уравнений первого порядка в полярных координатах. Для решения этой системы используется метод алгебраизации. В этом подходе полная 15-компонентная волновая функция раскладывается на сумму трех частей. Зависимость компонент в каждой части определяется только одной функцией $F_i(r), i = 1, 2, 3$. Три основные переменные построены в терминах вырожденных гипергеометрических функций. При этом существует правило квантования для некоторого спектрального параметра. Дополнительно возникает алгебраическая однородная система из 15 уравнений, которая полностью определяет структуру 15-компонентных решений. Из обращения в нуль определителя этой линейной системы получено кубическое алгебраическое уравнение относительно параметра энергии ε^2. Его решения найдены в аналитическом виде и исследованы численно. Получено три энергетических спектра, один из которых тривиален и не зависит от параметра поляризуемости, а два других существенно модифицированы этой дополнительной характеристикой.

Ключевые слова:

частица со спином 1, поляризуемость, цилиндрическая симметрия, внешнее однородное магнитное поле, разделение переменных, метод алгебраизации, точные решения, энергетические спектры

Introduction

In the present paper, we will study a vector particle with additional electromagnetic characteristics – polarizability in presence of the external uniform magnetic field. We specify the relevant 15-component equation [1–6] to cylindrical coordinates and tetrad. After separating the variables, we derive the system of 15 first-order differential equations in polar coordinate. To resolve this system, we apply the algebraization method. Within this approach, the complete 15-component wave function is presented as a sum of three constituents. Dependence of each constituent on the polar coordinates is determined by only one function $F_i(r), i = 1, 2, 3$. We find expressions for these variables $F_i(r)$ in terms of the confluent hypergeometric functions. At this there arises a quantization rule due to the presence of the external magnetic field. Besides, there is an algebraic homogenous system of 15 equations. Its solutions determine the structure of 15-component solutions. From vanishing the determinant of the system, it follows a third-order equation with respect to the energy parameter ε^2. Its solutions are found in analytic form and studied numerically. So, we have found three series of physically interpretable energy levels. One does not depend on the polarizability parameter σ and the other two are substantially modified by the polarizability parameter σ.

1. The basic equation

Initial equation has the form [7]

$$\left[\Gamma^{\alpha}(x)\left(\partial_{\alpha} + B_{\alpha} - i\frac{e}{\hbar}A_{\alpha}\right) - i\frac{Mc}{\hbar}\right]\Psi(x) = 0,$$

where

$$\Gamma^{\alpha}(x) = \Gamma^{a} e_{(a)}^{\alpha}(x), \quad B_{\alpha}(x) = \frac{1}{2}J^{ab} e_{(a)}^{\beta} \nabla_{\alpha} e_{(b)\beta},$$

$e_{(a)}^{\alpha}(x)$ is a tetrad, J^{ab} represents generators of the used 15-component set of tensors. Below we will use the notation M instead of $Mc/\hbar$. To the uniform magnetic field $\vec{B} = (0, 0, B)$, there corresponds the following 4-potential

$$A_0 = 0,\ A_r = 0,\ A_{\phi} = -\frac{Br^2}{2},\ A_z = 0.$$

We will use the diagonal cylindrical tetrad and the relevant Ricci rotation coefficients

$$\gamma_{ab0}(x) = 0, \quad \gamma_{ab1}(x) = 0,$$

$$\gamma_{ab2}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & -1/r & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \gamma_{ab3}(x) = 0.$$

The local matrices $\Gamma^{\alpha}(x)$ have the form

$$\Gamma^{\alpha}(x) = (\Gamma^{t}(x), \Gamma^{r}(x), \Gamma^{\phi}(x), \Gamma^{z}(x)) =$$

$$\left(\Gamma^0, \Gamma^1, \frac{\Gamma^2}{r}, \Gamma^3\right).$$

In this way, we get the following explicit form of the basic free equation

$$\Big[\Gamma^0\partial_0 + \Gamma^1\partial_r + \frac{1}{r}\Gamma^2(\partial_{\phi} + J^{12}) +$$

$$+ \Gamma^3\partial_z - M\Big]\Psi(t, r, \phi, z) = 0. \qquad (1)$$

The presence of the magnetic field may be taken into account by using the formal change in eq. (1):

$$\frac{1}{r}\Gamma^2(\partial_{\phi} + J^{12}) \Rightarrow$$

$$\Rightarrow \frac{1}{r}\Gamma^2\left(\partial_{\phi} + \frac{ieB}{2\hbar}r^2 + J^{12}\right).$$

We will take into account the last replacement later on, besides, we will apply the shortening notation B instead of $eB/(2\hbar)$.

For separating the variables, we need the explicit form of the matrices Γ^{α}. The most convenient is their representation in the so called cyclic basis, where the generator J^{12} is diagonal (we will apply the block structure of the matrices with dimensions $1, 1, 3, 1, 3, 3, 3$, see in [7]):

$$\Gamma^0 = \begin{pmatrix} 0 & 1 & \vec{0} & 0 & \vec{0} & \vec{0} & 0 \\ 1 & 0 & \vec{0} & 0 & \vec{0} & \vec{0} & \vec{0} \\ \vec{0}^t & \vec{0}^t & 0 & \vec{0}^t & 0 & -\sigma I & 0 \\ 1 & 0 & \vec{0} & 0 & \vec{0} & \vec{0} & \vec{0} \\ \vec{0}^t & \vec{0}^t & 0 & \vec{0}^t & 0 & 0 & 0 \\ \vec{0}^t & \vec{0}^t & 0 & \vec{0}^t & \pm I & 0 & 0 \\ \vec{0}^t & \vec{0}^t & 0 & \vec{0}^t & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^t = \begin{pmatrix} 0 & 0 & -\vec{e}_i & 0 & \vec{0} & \vec{0} & \vec{0} \\ 0 & 0 & \vec{0} & 0 & \vec{0} & -\sigma\vec{e}_i & \vec{0} \\ \vec{e}_i^{\,t} & \vec{0}^t & 0 & \vec{0}^t & 0 & 0 & -\sigma\tau_i \\ 0 & 0 & \vec{0} & 0 & \vec{0} & \vec{0} & \vec{0} \\ \vec{e}_i^{\,t} & \vec{0}^t & 0 & \vec{0}^t & 0 & 0 & 0 \\ \vec{0}^t & \vec{0}^t & 0 & -(\pm)\vec{e}_i^{\,t} & 0 & 0 & 0 \\ \vec{0}^t & \vec{0}^t & 0 & \vec{0}^t & (\pm)\tau_i & 0 & 0 \end{pmatrix},$$

where $\vec{e}_i, \vec{e}_i^{\,t}, \tau$ stand for the matrices

$$\vec{e}_1 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\vec{e}_2 = \left(-\frac{i}{\sqrt{2}}, 0, -\frac{i}{\sqrt{2}}\right),\ \vec{e}_3 = (0, 1, 0),$$

$$\vec{e}_1^{\,t} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix},\ \vec{e}_2^{\,t} = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \end{pmatrix},\ \vec{e}_3^{\,t} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\tau_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{pmatrix},\ \tau_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\tau_3 = -i\begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

the generator J^{12} is diagonal, $J^{12} = \text{diag}(0, 0, \tau_3, 0, \tau_3, \tau_3, \tau_3)$.

2. Separating the variables

Let us search for solutions in the form of cylindrical waves:

$$\Psi\{C,C_0,\vec{C},\Phi_0,\vec{\Phi},\vec{E},\vec{H}\},\ C(x)=e^{-i\epsilon t}e^{ikz}e^{im\phi}C(r),$$

$$C_0(x)=e^{-i\epsilon t}e^{ikz}e^{im\phi}C_0(r),$$

$$\vec{C}(x)=e^{-i\epsilon t}e^{ikz}e^{im\phi}\begin{pmatrix}C_1(r)\\C_2(r)\\C_3(r)\end{pmatrix},$$

$$\Phi_0(x)=e^{-i\epsilon t}e^{ikz}e^{im\phi}\Phi_0(r),$$

$$\vec{\Phi}(x)=e^{-i\epsilon t}e^{ikz}e^{im\phi}\begin{pmatrix}\Phi_1(r)\\\Phi_2(r)\\\Phi_3(r)\end{pmatrix},$$

$$\vec{E}(x)=e^{-i\epsilon t}e^{ikz}e^{im\phi}\begin{pmatrix}E_1(r)\\E_2(r)\\E_3(r)\end{pmatrix},$$

$$\vec{H}(x)=e^{-i\epsilon t}e^{ikz}e^{im\phi}\begin{pmatrix}H_1(r)\\H_2(r)\\H_3(r)\end{pmatrix}.$$

Taking in mind the block structure of the wave function and matrices, we obtain the block equations

$$-i\frac{\epsilon}{\hbar c}C_0-\vec{e}_1\frac{d}{dr}\vec{C}-\frac{1}{r}\vec{e}_2(im+\tau_3)\vec{C}-ik\vec{e}_3\vec{C}=MC,$$

$$-\vec{e}_1\frac{d}{dr}\vec{E}-\frac{1}{r}\vec{e}_2(im+\tau_3)\vec{E}-ik\vec{e}_3\vec{E}=MC_0,$$

$$i\frac{\epsilon}{\hbar c}-\tau_1\frac{d}{dr}\vec{H}-\frac{1}{r}\tau_2(im+\tau_3)\vec{H}-ik\tau_3\vec{H}=M\vec{C},$$

$$-i\frac{\epsilon}{\hbar c}\sigma C-\frac{1}{r}\vec{e}_2(im+\tau_3)\vec{E}-ik\vec{e}_3\vec{E}=M\Phi_0,$$

$$i\frac{\epsilon}{\hbar c}\vec{E}+\sigma\vec{e}_1^{\,t}\frac{d}{dr}C-\tau_1\frac{d}{dr}\vec{H}+$$

$$+\frac{1}{r}\left[im\sigma\vec{e}_2^{\,t}C-\tau_2(im+\tau_3)\vec{H}\right]+$$

$$+ik(\sigma\vec{e}_3^{\,t}C-\tau_3\vec{H})=M\vec{\Phi},$$

$$-i\frac{\epsilon}{\hbar c}\vec{\Phi}-\vec{e}_1^{\,t}\frac{d}{dr}\Phi_0-\frac{1}{r}im\vec{e}_2^{\,t}\Phi_0-ik\vec{e}_3^{\,t}\Phi_0=M\vec{E},$$

$$\tau_1\frac{d}{dr}\vec{\Phi}+\frac{1}{r}\tau_2(im+\tau_3)\vec{\Phi}+ik\tau_3\vec{\Phi}=M\vec{H}.$$

With the use of the shortening notations (to take into account the presence of the magnetic field, we make the change $m\Rightarrow m+Br^2$) :

$$m\frac{1}{\sqrt{2}}\left(\frac{d}{dr}+\frac{m+Br^2}{r}\right)=a_m,$$

$$\frac{1}{\sqrt{2}}\left(-\frac{d}{dr}+\frac{m+Br^2}{r}\right)=b_m,$$

we can present the resulting system of equations as follows

$$-i\epsilon C_0-b_{m-1}C_1-a_{m+1}C_3-ikC_2=MC;$$

$$-b_{m-1}E_1-a_{m+1}E_3-ikE_2=MC_0,$$

$$i\epsilon E_1+ia_mH_2-kH_1=MC_1,$$

$$i\epsilon E_2-ib_{m-1}H_1+ia_{m+1}H_3=MC_2;$$

$$i\epsilon E_3-ib_mH_2+kH_3=MC_3,$$

$$-i\epsilon\sigma C-b_{m-1}E_1-a_{m+1}E_3-ikE_2=M\Phi_0,$$

$$i\epsilon E_1-\sigma a_mC+ia_mH_2-kH_1=M\Phi_1,$$

$$i\epsilon E_2-ib_{m-1}H_1+ia_{m+1}H_3+ik\sigma C=M\Phi_2,$$

$$i\epsilon E_3-\sigma b_mC-ib_mH_2+kH_3=M\Phi_3;$$

$$-i\epsilon\Phi_1+a_m\Phi_0=ME_1,\quad -i\epsilon\Phi_2-ik\Phi_0=ME_2,$$

$$-i\epsilon\Phi_3+b_m\Phi_0=ME_3,\quad -ia_m\Phi_2+k\Phi_1=MH_1,$$

$$ib_{m-1}\Phi_1-ia_{m+1}\Phi_3=MH_2,\ ib_m\Phi_2-k\Phi_3=MH_3.$$

3. Solving the system of equations, algebraization method

Analyzing the structure of the above equations, we can notice that they may be transformed to algebraic form, if the complete wave function is presented as the sum of three parts and each part is determined by only one corresponding function $F_1(r),F_2(r),F_3(r)$:

$$\Psi=\Psi_1(r)+\Psi_2(r)+\Psi_3(r),$$

$$\Psi_1(r)\Big(C(r),C_0(r),C_1(r),C_2(r),C_3(r),\Phi_0(r),\Phi_1(r),$$

$$\Phi_2(r),\Phi_3(r),E_1(r),E_2(r),E_3(r),H_1(r),H_2(r),H_3(r)\Big)^t=$$

$$=\Big(C,C_0,0,C_2,0,\Phi_0,0,\Phi_2,0,0,E_2,0,0,H_2,0\Big)^tF_1(r)+$$

$$+\Big(0,0,C_1,0,0,0,\Phi_1,0,0,E_1,0,0,H_1,0,0\Big)^tF_2(r)+$$

$$+\Big(0,0,0,0,C_3,0,0,0,\Phi_3,0,0,E_3,0,0,H_3\Big)^tF_3(r),$$

where ^{t} denotes transposition. Additionally the following differential constraints should be imposed

$$b_{m-1}F_2(r)=\alpha_1F_1(r),\quad a_mF_1(r)=\alpha_3F_2(r),$$

$$a_{m+1}F_3(r)=\alpha_2F_1,\quad b_mF_1(r)=\alpha_4F_3(r),$$

where $\alpha_1,\ldots,\alpha_4$ are some numerical parameters. In the last constraints, without loss of generality, we can set $\alpha_3=\alpha_1$, $\alpha_4=\alpha_2$. Then we get

$$b_{m-1}F_2(r)=\alpha_1F_1(r),\quad a_mF_1(r)=\alpha_1F_2(r),$$

$$a_{m+1}F_3(r)=\alpha_2F_1,\quad b_mF_1(r)=\alpha_2F_3(r),$$

whence follow the second-order equations for the separate functions:

$$(b_{m-1}a_m-\alpha_1^2)F_1(r)=0,\ (a_mb_{m-1}-\alpha_1^2)F_2(r)=0,$$

$$(a_{m+1}b_m-\alpha_2^2)F_1(r)=0,\ (b_ma_{m+1}-\alpha_2^2)F_3(r)=0.$$

We will turn to these equations later on.

In this way, we arrive at the following algebraic system

$$-i\epsilon C_0 - \alpha_1 C_1 - ikC_2 - \alpha_2 C_3 = MC,$$

$$-\alpha_1 E_1 - ikE_2 - \alpha_2 E_3 = MC_0,$$

$$i\epsilon E_1 - ikH_1 + i\alpha_1 H_2 = MC_1,$$

$$i\epsilon E_2 - i\alpha_1 H_1 + i\alpha_2 H_3 = MC_2,$$

$$i\epsilon E_3 - i\alpha_2 H_2 + kH_3 = MC_3,$$

$$-i\epsilon\sigma C - \alpha_1 E_1 - ikE_2 - \alpha_2 E_3 = M\Phi_0,$$

$$-\sigma\alpha_1 C + i\epsilon E_1 - kH_1 + i\alpha_1 H_2 = M\Phi_1,$$

$$ik\sigma C + i\epsilon E_2 - i\alpha_1 H_1 + i\alpha_2 H_3 = M\Phi_2,$$

$$-\sigma\alpha_2 C + i\epsilon E_3 - i\alpha_2 H_2 + kH_3 = M\Phi_3,$$

$$\alpha_1\Phi_0 - i\epsilon\Phi_1 = ME_1,\ -ik\Phi_0 - i\epsilon\Phi_2 = ME_2,$$

$$\alpha_2\Phi_0 - i\epsilon\Phi_3 = ME_3,\ k\Phi_1 - i\alpha_1\epsilon\Phi_2 = MH_1,$$

$$i\alpha_1\Phi_1 - i\alpha_2\Phi_3 = MH_2,\ i\alpha_2\Phi_2 - k\Phi_3 = MH_3.$$

It may be presented in the matrix form $A_{15\times15}\Psi = 0$, where matrix $A_{15\times15}$ is as follows

$$A_{15\times15} =$$

$$= \begin{pmatrix}
-M & -i\epsilon & -\alpha_1 & -ik & -\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 & -ik & -\alpha_2 & 0 & 0 & 0 \\
0 & 0 & -M & 0 & 0 & 0 & 0 & 0 & 0 & i\epsilon & 0 & 0 & -k & i\alpha_1 & 0 \\
0 & 0 & 0 & -M & 0 & 0 & 0 & 0 & 0 & 0 & i\epsilon & 0 & -i\alpha_1 & 0 & i\alpha_2 \\
0 & 0 & 0 & 0 & -M & 0 & 0 & 0 & 0 & 0 & 0 & i\epsilon & 0 & -i\alpha_2 & k \\
-i\epsilon\sigma & 0 & 0 & 0 & 0 & -M & 0 & 0 & 0 & -\alpha_1 & -ik & -\alpha_2 & 0 & 0 & 0 \\
-\sigma\alpha_1 & 0 & 0 & 0 & 0 & 0 & -M & 0 & 0 & i\epsilon & 0 & 0 & -k & i\alpha_1 & 0 \\
ik\sigma & 0 & 0 & 0 & 0 & 0 & 0 & -M & 0 & 0 & i\epsilon & 0 & -i\alpha_1 & 0 & i\alpha_2 \\
-\sigma\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -M & 0 & 0 & i\epsilon & 0 & -i\alpha_2 & k \\
0 & 0 & 0 & 0 & 0 & \alpha_1 & -i\epsilon & 0 & 0 & -M & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -ik & 0 & -i\epsilon & 0 & 0 & -M & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & -i\epsilon & 0 & 0 & -M & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & k & -i\alpha_1 & 0 & 0 & 0 & 0 & -M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i\alpha_1 & 0 & -i\alpha_2 & 0 & 0 & 0 & 0 & -M & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i\alpha_2 & -k & 0 & 0 & 0 & 0 & 0 & -M
\end{pmatrix}.$$

From vanishing its determinant, we get

$$\det A = -M^5(\alpha_1^2 + \alpha_2^2 + k^2 + M^2 - \epsilon^2)\times$$

$$\times\Big\{\alpha_1^6\sigma(k^2 + M^2 - \epsilon^2) + \alpha_1^4(k^2 + M^2 - \epsilon^2)\times$$

$$\times\left[-\alpha_2^2\sigma + \sigma(k^2 - \epsilon^2) + M^2(\sigma + 1)\right] -$$

$$-\alpha_1^2\big\{\alpha_2^4(k^2 + M^2 - \epsilon^2) - 2M^4(k^2 + M^2 - \epsilon^2) +$$

$$+2\alpha_2^2\left[k^2(M^2(2\sigma + 1) - 2\sigma\epsilon^2) + k^4\sigma +\right.$$

$$\left.+M^4(\sigma - 1) - M^2(2\sigma + 1)\epsilon^2 + \sigma\epsilon^4\right]\big\} +$$

$$+(k^2+M^2-\epsilon^2)\left[\alpha_2^6\sigma + \alpha_2^4(\sigma(k^2 - \epsilon^2) + M^2(\sigma + 1)) +\right.$$

$$\left.+M^4(k^2 + M^2 - \epsilon^2) + 2\alpha_2^2 M^4\right]\Big\} = 0.$$

Further we find three roots:

$$\epsilon_0^2 = \alpha_1^2 + \alpha_2^2 + k^2 + M^2,$$

$$\epsilon_\pm^2 = \frac{1}{2[M^4 + \sigma(\alpha_1^4 + \alpha_2^4 - 2\alpha_1^2\alpha_2^2)]}\times$$

$$\times\Big\{\sigma[(\alpha_1^6 + \alpha_2^6 - \alpha_1^2\alpha_2^4 - \alpha_1^4\alpha_2^2) +$$

$$+(M^2 + k^2)(2\alpha_1^4 + 2\alpha_2^4 - 4\alpha_1^2\alpha_2^2)] +$$

$$+(2M^6 + 2\alpha_1^2 M^4 + 2\alpha_2^2 M^4 + \alpha_1^4 M^2 +$$

$$+\alpha_2^4 M^2 - 2\alpha_1^2\alpha_2^2 M^2 + 2k^2 M^4)\pm$$

$$\pm(\alpha_1^2 - \alpha_2^2)[\sigma^2(\alpha_1^8 + \alpha_2^8 - 2\alpha_1^4\alpha_2^4) +$$

$$+\sigma(4\alpha_1^4 M^4 + 4\alpha_2^4 M^4 - 8\alpha_1^2\alpha_2^2 M + 2\alpha_1^6 M^2 +$$

$$+2\alpha_2^6 M^2 - 2\alpha_1^2\alpha_2^4 M^2 - 2\alpha_1^4\alpha_2^2 M^2) +$$

$$+(4M^8 + 4\alpha_1^2 M^6 + 4\alpha_2^2 M^6 +$$

$$+\alpha_1^4 M^4 + \alpha_2^4 M^4 - 2\alpha_1^2\alpha_2^2 M^4)]^{1/2}\Big\}. \quad (2)$$

4. Second-order equations for basic functions

Let us turn to the second-order equations for separate functions. Taking in mind the definitions

$$a_m = \frac{1}{\sqrt{2}}\left(\frac{d}{dr} + \frac{m + Br^2}{r}\right),$$

$$b_m = \frac{1}{\sqrt{2}}\left(-\frac{d}{dr} + \frac{m + Br^2}{r}\right),$$

we get their explicit form

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - B^2r^2 - \right.$$

$$\left. -\frac{m^2}{r^2} - 2Bm + 2B + 2\alpha_1^2\right] F_1 = 0,$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - B^2r^2 - \right.$$

$$\left. -\frac{m^2}{r^2} - 2Bm - 2B + 2\alpha_2^2\right] F_1 = 0,$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - B^2r^2 - \right.$$

$$\left. -\frac{(m-1)^2}{r^2} - 2Bm + 2\alpha_1^2\right] F_2 = 0,$$

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-B^2r^2- -\frac{(m+1)^2}{r^2}-2Bm+2\alpha_2^2\right]F_3=0.$$

From two first equations, it follows a constraint for α_1^2 and α_2^2:

$$2\alpha_2^2=2\alpha_1^2+4B.$$

Let us introduce the notation $2B+2\alpha_1^2=X$, then the above two equations read

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-B^2r^2-\frac{m^2}{r^2}-2Bm+X\right]F_1=0.$$

Correspondingly, two remaining equations are presented as follows

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-B^2r^2-\frac{(m-1)^2}{r^2}- -2B(m+1)+X\right]F_2=0,$$

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-B^2r^2-\frac{(m+1)^2}{r^2}- -2B(m-1)+X\right]F_3=0.$$

We can notice the symmetry between them: $m\Leftrightarrow -m$, $B\Leftrightarrow -B$.

Now, we take the inverse transformation $B/2\Leftarrow B$ (see (2)); then we obtain three different equations:

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{B^2r^2}{4}-\frac{m^2}{r^2}-Bm+X\right]F_1=0,$$

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{B^2r^2}{4}-\frac{(m-1)^2}{r^2}- -B(m+1)+X\right]F_2=0,$$

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{B^2r^2}{4}-\frac{(m+1)^2}{r^2}- -B(m-1)+X\right]F_3=0.$$

In the variable $x=Br^2/2$, they read more symmetrically:

$$\left[\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{1}{4}-\frac{(m/2)^2}{x^2}+ +\frac{1}{x}\left(-\frac{m}{2}+\frac{X}{2B}\right)\right]F_1=0,$$

$$\left[\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{1}{4}-\frac{[(m+1)/2]^2}{x^2}+ +\frac{1}{x}\left(-\frac{m-1}{2}+\frac{X}{2B}\right)\right]F_2=0,$$

$$\left[\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{1}{4}-\frac{[(m-1)/2]^2}{x^2}+ +\frac{1}{x}\left(-\frac{m+1}{2}+\frac{X}{2B}\right)\right]F_3=0.$$

These three equations are of the same type. It is enough to examine the first one, and the similar results for the remaining two can be obtained by formal changes. We are looking for solutions of first equation in the form $F_1(x)=x^Ae^{Cx}f_1(x)$. For function $f_1(x)$ we get the equation

$$xf_1^{''}+(2A+1+2Cx)f_1'+\left[\frac{1}{x}(A^2-(m/2)^2)+ +\left(2AC+C-\frac{m}{2}+\frac{X}{2B}\right)+x\left(C^2-\frac{1}{4}\right)\right]f_1=0.$$

Let

$$A^2-(m/2)^2=0\Rightarrow A=\pm|m/2|,$$

$$C^2-\frac{1}{4}=0\Rightarrow C=\pm\frac{1}{2}.$$

To construct solutions related to bound states, we are to set $A=+|m/2|$, $C=-1/2$, then the above equation simplifies

$$xf_1^{''}+(|m|+1-x)f_1'-\left(\frac{|m|+m}{2}+\frac{1}{2}-\frac{X}{2B}\right)f_1=0.$$

It is of confluent hypergeometric type with parameters

$$a=\frac{|m|+m}{2}+\frac{1}{2}-\frac{X}{2B},\quad c=|m|+1,$$

$$F=\Phi(a,c,x).$$

The polynomial condition $a=-n_1$ gives

$$X=+2B\left(\frac{|m|+m}{2}+\frac{1}{2}+n_1\right)>0,\ n_1=0,1,2,\ldots.$$

This spectrum corresponds to the following solutions:

$$F_1(x)=x^{+\frac{|m|}{2}}e^{-x/2}f_1(x),$$

$$f_1(x)=\Phi(-n_1,|m|+1,x).$$

Other two equations lead to similar results. Thus we have

$$F_1(x)=x^{+\frac{|m|}{2}}e^{-x/2}f_1(x),$$

$$f_1(x)=\Phi(-n_1,|m|+1,x).$$

$$X=2B\left(\frac{|m|+m}{2}+\frac{1}{2}+n_1\right)>B,$$

$$n_1=0,1,2,\ldots;$$

$$F_2(x)=x^{+\frac{|m+1|}{2}}e^{-x/2}f_2(x),$$

$$f_2(x)=\Phi(-n_2,|m+1|+1,x),$$

$$X=2B\left(\frac{|m+1|+m-1}{2}+\frac{1}{2}+n_2\right)>B,$$

$$n_2=0,1,2,\ldots;$$

$$F_3(x)=x^{+\frac{|m-1|}{2}}e^{-x/2}f_3(x),$$

$$f_3(x) = \Phi(-n_3, |m-1|+1, x),$$

$$X = 2B\left(\frac{|m-1|+m+1}{2} + \frac{1}{2} + n_3\right) > B,$$

$$n_3 = 0, 1, 2, \ldots.$$

In all three cases the quantity X is the same, below we will apply the variant

$$X = 2BN > 0,$$

$$N = \left(\frac{|m|+m}{2} + \frac{1}{2} + n\right), \; n = 0, 1, 2, \ldots.$$

Note that parameter N takes half-integer values. Note the formulas

$$X = B + 2\alpha_1^2,$$

$$2\alpha_2^2 = 2B + 2\alpha_1^2 = 2B + (X - B) = B + X,$$

$$\alpha_1 = \frac{1}{\sqrt{2}}\sqrt{X - B}, \quad \alpha_2 = \frac{1}{\sqrt{2}}\sqrt{X + B}, \qquad (3)$$

$$X = 2BN, \quad N = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots.$$

5. Numerical study of the energy spectra

Taking into account relations (3), we transform equation $\det A = 0$ to the more simple form (it is factorized into two equations)

$$k^2 + M^2 + X - \epsilon^2 = 0,$$

$$\epsilon^4\Big(B^2\sigma + M^4\Big) + \epsilon^2\Big(-2B^2k^2\sigma - B^2M^2(2\sigma+1) -$$

$$-B^2\sigma X - 2k^2M^4 - 2M^4X - 2M^6\Big) +$$

$$+B^2k^2M^2(2\sigma+1) + B^2k^4\sigma + B^2k^2\sigma X +$$

$$+B^2M^4\sigma + B^2M^2\sigma X + 2k^2M^4X + 2k^2M^6 +$$

$$+k^4M^4 + M^4X^2 + 2M^6X + M^8 = 0.$$

In dimensionless quantities

$$\frac{\epsilon^2}{M^2} = E^2, \quad \frac{B}{M^2} = b,$$

$$\frac{k^2}{M^2} = K^2, \quad \frac{X}{M^2} = x = 2bN,$$

they read simpler:

$$\epsilon^2 = K^2 + 1 + x,$$

and

$$E^4\Big(b^2\sigma + 1\Big) + E^2\Big(-2b^2K^2\sigma - b^2(2\sigma+1) -$$

$$b^2\sigma x - 2K^2 - 2x - 2\Big) + b^2K^2(2\sigma+1) +$$

$$+b^2K^4\sigma + b^2K^2\sigma x + b^2\sigma + b^2\sigma x +$$

$$+2K^2x + 2K^2 + K^4 + x^2 + 2x + 1 = 0.$$

The second equation leads to the roots

$$E^2 = \frac{1}{2(1+b^2\sigma)} \times$$

$$\times\Big[b^2 + 2K^2 + 2x + 2 + (2b^2K^2 + 2b^2 + b^2x)\sigma \pm$$

$$\pm b\sqrt{b^2 + 4x + 4 + (4b^2\sigma + 2b^2\sigma x + b^2\sigma^2x^2)}\Big],$$

$$x = 2bN, \quad N = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots.$$

Let us detail numerically several particular cases:

$b = 0.1, K = 0.1, \sigma = 0.01$

	ϵ_0^2	ϵ_-^2	ϵ_+^2
$N = 1/2$	1.05357	1.01	1.21999
$N = 3/2$	1.14455	1.20086	1.42911
$N = 5/2$	1.22882	1.39241	1.63754
$N = 7/2$	1.30767	1.58449	1.84544
$N = 9/2$	1.38203	1.77703	2.05288
$N = 11/2$	1.45258	1.96995	2.25994
$N = 13/2$	1.51987	2.1632	2.46666
$N = 15/2$	1.5843	2.35674	2.67311

$b = 1, K = 0.1, \sigma = 0.01$

	ϵ_0^2	ϵ_-^2	ϵ_+^2
$N = 1/2$	1.41774	1.01	3.9902
$N = 3/2$	2.0025	2.44301	6.53739
$N = 5/2$	2.45153	3.99801	8.96259
$N = 7/2$	2.83019	5.6186	11.3222
$N = 9/2$	3.16386	7.28183	13.6392
$N = 11/2$	3.46554	8.97564	15.9255
$N = 13/2$	3.74299	10.6928	18.1886
$N = 15/2$	4.00125	12.4285	20.4331

$b = 0.1, K = 0.1, \sigma = 0.001$

	ϵ_0^2	ϵ_-^2	ϵ_+^2
$N = 1/2$	1.05357	1.01	1.22
$N = 3/2$	1.14455	1.20087	1.42912
$N = 5/2$	1.22882	1.39242	1.63757
$N = 7/2$	1.30767	1.58452	1.84548
$N = 9/2$	1.38203	1.77707	2.05293
$N = 11/2$	1.45258	1.97	2.25999
$N = 13/2$	1.51987	2.16325	2.46673
$N = 15/2$	1.5843	2.3568	2.67318

$b = 1, K = 0.1, \sigma = 0.001$

	ϵ_0^2	ϵ_-^2	ϵ_+^2
$N = 1/2$	1.41774	1.01	4.008
$N = 3/2$	2.0025	2.4479	6.5681
$N = 5/2$	2.45153	4.0088	9.00521
$N = 7/2$	2.83019	5.63581	11.3762
$N = 9/2$	3.16386	7.30578	13.7042
$N = 11/2$	3.46554	9.00657	16.0014
$N = 13/2$	3.74299	10.7309	18.2752
$N = 15/2$	4.00125	12.4738	20.5302

Conclusions

We can see that only two obtained spectra depend on the value of additional parameters σ.

Besides, it should be noted that the used algebraization method to solve the system of 15 differential equations is more simple than previously applied method for treating this problem in the paper [7], though the energy spectra are the same.

Also it should be mentioned that the used algebraization method is closely related to method developed by Gronskiy and Fedorov [8] for treating the particle with multi-spins $S = 1/2 \oplus 3/2$, where the three constituents of the complete wave function were fixed with the use of projective operators constructed from generator J^{12} of the field. The algebraization method may be effectively used for separating the variables in various physical problems.

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